NOTES AND RECORDS

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THE LIFE AND WORK OF ROBERT J. ADCOCK. PART I: THE FIGURE OF THE EARTH

by

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The American mathematician Robert Jackson Adcock (1826–1895) is an obscure figure, hitherto associated with the history of regression analysis and least-squares, whose identity and life are described in Part II of this work. In 1872, he self-published a pamphlet, 'Gravitation to the sphere and the two ellipsoids of revolution: ratio of the axes of a rotating fluid mass', which seems to have been largely ignored at the time. It effectively became lost thereafter, until a copy was recently discovered in the Library of the Royal Society. In it, he determined the degree of flattening assumed by a rotating homogeneous *ellipsoidal* fluid Earth with a uniform density, subject to gravitational attraction and in hydrostatic equilibrium—a problem previously considered, but not solved, by Newton, Laplace, Gauss and Dirichlet. Adcock successfully obtained an explicit solution for the potential of a homogeneous ellipsoid and correctly calculated the flattening of such a model. Until now, this result was believed to have first been obtained by a German engineer, Otto Heymann, in 1935. Adcock's pamphlet is transcribed here with a commentary on its contents. It is hoped that recognition of his remarkable achievement will enhance his reputation as a mathematician.

Keywords: figure of the Earth; dynamics; potential theory; physical astronomy; history of mathematics

INTRODUCTION

For many years, the name of the American mathematician R. J. Adcock has been associated with the history of regression, specifically, fitting a straight-line relationship, $y = \alpha + \beta x$, when both x and y are subject to error, and ideas associated with the eventual development

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of principal components analysis.[1] An account of his life, so far as its details can now be ascertained, is given in Part II of this work.

The majority of his mathematical publications were short notes, mathematical conundrums, or solutions to his own problems or to those set by others. These appeared mainly in the journal *The Analyst* (1874–1883), but he also published in the *Annals of Mathematics*, the *American Mathematical Monthly*, the *Astronomical Journal* and the *Sidereal Messenger*. Unfortunately, his contributions, including the self-published work discussed here, were sometimes marred by algebraic or arithmetic errors, either of his own making[2] or attributed to the compositor.[3]

Gravitational attraction and the figure of the Earth

In the late seventeenth century, the English scientist Robert Hooke (1635–1703) was thinking about the orbital dynamics of planetary motions and concluded that:

> All Coelestial Bodies whatsoever, have an attraction or gravitating power towards their own Centres, whereby they attract not only their own parts, and keep them from flying from them, as we may observe the Earth to do, but that they do also attract all the other Coelestial bodies that are within the sphere of their activity.[4]

Similar ideas had been considered by earlier investigators, including the French astronomer Ismaël Bullialdus (1605–1694), the Italian physicist Giovanni Borelli (1608–1679) and Hooke's fellow-countryman Isaac Newton (1643–1727), but it was Hooke's suggestion 'that all bodies whatsoever that are put into a direct and simple motion, will so continue to move forward in a streight line, till they are by some other effectual powers deflected and bent into a Motion, describing a Circle, Ellipsis, or some other more compounded Curve Line' which he applied to planetary motion around the Sun, believing that gravitational attraction originated in periodic pulses (analogous to light and sound).[5]

Hooke's notion of compounding the tangential velocity of a body with a radial velocity resulting from its attraction to a central body (such as the Sun) proved immensely helpful to Newton's consideration of the mechanisms of celestial dynamics. In a subsequent letter to Newton in January 1679, Hooke revealed that he supposed that 'the Attraction always is in a duplicate proportion to the Distance from the Centre Reciprocall', i.e. that the force of gravity is inversely proportional to the square of distance.[6] By so doing, he precipitated a priority dispute with Newton who, by 1669, had arrived at the same idea independently from a sound theoretical basis, although he had not published it.[7] However, in Newton's

1 R. J. Adcock, 'The correct method of least squares', *Analyst* **1**, 64 (1874); R. J. Adcock, 'Note on the method of least squares', *Analyst* **4**, 183–184 (1877); R. J. Adcock, 'Least squares', *Analyst* **5**, 21–22 (1878); R. J. Adcock, 'A problem in least squares', *Analyst* **5**, 53–54 (1878); R. J. Adcock, 'Extension of the method of least squares to any number of variables', *Analyst* **7**, 22–23 (1880).

2 Adcock, 'A problem', *op. cit.* (note 1), p. 54, equation 5.

3 R. J. Adcock, 'Solution' [to Problem 239], *Analyst* **6**, 49 (1879).

4 Robert Hooke, *An Attempt to Prove the Motion of the Earth from Observations* (J. Martyn, London, 1674), pp. 27–28.

5 *Ibid.*, p. 28.

6 The letter is reproduced as document 239 in Hubert Westren Turnbull, J. F. Scott, A. Rupert Hall and Laura Tilling (eds), *The correspondence of Isaac Newton, vol. 2: 1676–1687* (Cambridge University Press, Cambridge, 1960), p. 309. Hooke, primarily a superb experimentalist, lacked a sufficient depth of mathematics compared to Newton and this was probably no more than an inspired guess.

7 See Michael Nauenberg, 'Robert Hooke's seminal contribution to orbital dynamics', *Phys. Perspect.* **7**, 4–34 (2005).

Philosophiae naturalis principia mathematica (*Mathematical principles of natural philosophy*), book 1, proposition 76, theorem 36, he states:

> If spheres are in any way nonhomogeneous (as to the density of their matter and their attractive force) going from the center to the circumference, but are uniform throughout in every spherical shell at any given distance from the center, and the attractive force of each point decreases in the squared ratio of the distance of the attracted body, I say that the total force by which one sphere of this sort attracts another is inversely proportional to the square of the distance between their centres. … Hence, if many spheres of this sort, similar to one another in all respects, attract one another, the accelerative attraction of any one to any other of them, at any equal distances between the centres, will be as the attracting spheres. And at any unequal distances, as the attracting sphere divided by the square of the distances between the centers.[8]

As shown by Chandrasekhar, later in this proposition Newton proves that the (gravitational) attractive force F between any pair of spheres i and j, with masses M_i and M_j, separated by a distance R_{ij} is:

$$F(M_i, M_j) = \mathcal{C}\frac{M_i M_j}{R_{ij}^2}$$

where $\mathcal{C}$ is a constant of proportionality (which has since come to be known as Newton's gravitational constant).[9] Ideally, if $M_\oplus$ is the mass of the Earth, $R_\oplus$ is its mean radius, the mean gravitational acceleration at the Earth's surface is g and its mean density is $\rho_\oplus$, then:

$$\mathcal{C} = \frac{gM_\oplus}{R_\oplus^2} = \frac{3g}{4\pi R_\oplus \rho_\oplus}.$$

Although Newton had no ready means of determining its value, he foresaw that it ought to be possible to do so experimentally. However, subsequent investigators initially focused on $\rho_\oplus$. The first of these determinations was attempted by the French mathematician, geophysicist, geodesist and astronomer Pierre Bouguer (1698–1758) and the geographer and mathematician Charles Marie de la Condamine (1701–1744) at Chimborazo in Peru in

8 Isaac Newton, *The Principia: mathematical principles of natural philosophy. A new translation. Preceded by a guide to Newton's Principia* (ed. and trans. I. Bernard Cohen, Anne Whitman and Julia Budenz) (University of California Press, Berkeley and London, 1999), Book 1, Proposition 76, Theorem 36, pp. 595–596. See also Isaac Newton, *Philosophiae naturalis principia mathematica* (G. & J. Innys, London, 1687); Isaac Newton, *The Mathematical Principles of Natural Philosophy. Translated into English by Andrew Motte. To which are added, The Laws of the Moon's Motion according to Gravity By John Machin*, 2 vols in one (Benjamin Motte, London, 1729), vol. 2, pp. 272–274; Isaac Newton, *Newton's Principia: the mathematical principles of natural philosophy, by Sir Isaac Newton; translated into English by Andrew Motte. To which is added Newton's system of the world; with a portrait taken from the bust in the Royal Observatory at Greenwich. First American edition, carefully revised and corrected, with a life of the author, by N.W. Chittenden* (Daniel Adee, New York, 1846).

9 Subrahmanyan Chandrasekhar, *Newton's Principia for the common reader* (Clarendon Press, Oxford, 1995), at pp. 280–282; to be complemented by the original text edited by A. Koyré, I. B. Cohen and A. Whitman, *Isaac Newton's Philosophiae naturalis principia mathematica: the third edition (1726) with variant readings*, 2 vols (University Press, Cambridge, 1972); and the recent translation from Latin by Cohen *et al.*, *op. cit*. (note 8). Note that Cohen *et al.* say: 'the late S. Chandrasekhar's *Newton's Principia for the Common Reader* … is an essentially nonhistorical work by one of the world's foremost astrophysicists. Despite its title, it is written for readers who are well grounded in mathematics and celestial mechanics. Chandrasekhar basically devised his own proofs of propositions in books 1 and 3 and then compared his proofs with Newton's' (p. 295). Nevertheless, for the modern reader, the mathematics is considerably easier to understand when set out as equations rather than in Newton's prose-based style.

1737–1740. If g_h is the value of gravity measured on a plateau of height h above sea level, compared with a value of g_0 at sea level, and d is the density of the plateau then, ideally:

$$g_h = g_0\left(1 - \frac{2h}{R_\oplus} + \frac{3hd}{R_\oplus \rho_\oplus}\right).$$

The rocks forming the plateau provide d; $R_\oplus$ is determined from astronomical observations, and hence $\rho_\oplus$ may, in theory, be found. Unfortunately, their pendulum-based gravity measurements gave inconclusive results.[10]

The next attempt to measure the 'attraction of a mountain' was made at Schiehallion in Scotland in 1744–1746 by the Astronomer Royal, Neville Maskelyne (1732–1811), and the mathematician Charles Hutton (1737–1823), using a plumb-line.[11] This gave a mean density of 4.5–5 g cm^{-3}. Similar results followed, but in 1797–1798, the English natural philosopher Henry Cavendish (1731–1810), using a torsion balance newly invented by the Revd John Michell (1724–1793), who unfortunately died before he could use it, found the Earth's density to be 5.448 g cm^{-3} which corresponds to $\mathcal{C} = 6.74 \times 10^{-11}$ m^3 kg^{-1} s^{-2}.[12] The superiority in precision achievable by this approach over pendulum measurements was evident, and improved instruments of this type have continued to be used into modern times.[13] The present-day designation of the gravitational constant by G (often referred to as 'big G' to distinguish it from 'little g') was introduced by Charles Vernon Boys (1855–1944) in 1894, who found its value to be 6.658×10^{-11} m^3 kg^{-1} s^{-2}.[14]

In his discussion of proposition 76, Newton had shown that for an object in a circular orbit with a mean radius r around the Sun (or a planet) and a period T, if the central object is spherically symmetric and has a radius R, then the mean density of the object $\bar{\rho}$ is:

$$\bar{\rho} = k\left(\frac{r}{R}\right)^3 \frac{1}{T^2}$$

where the constant $k = \frac{3\pi}{G}$.[15] This tended to focus the aim of subsequent investigations on determining the density of the Earth.[16]

10 Pierre Bouguer, *La Figure de la Terre, Déterminée par les Observations de Messieurs Bouguer et de la Condamine … pour observer aux environs de l'Equateur. Avec une Relation abregée de ce Voyage, qui contient la description du Pays dans lequel les Opérations ont été faites* (Charles-Antoine Jombert, Paris, 1749).

11 Neville Maskelyne, 'An Account of Observations made on the Mountain Schehallien for finding its Attraction', *Phil. Trans. R. Soc. Lond.* **65**, 500–542 (1775); see also John R. Smallwood, 'Maskelyne's 1774 Schiehallion experiment revisited', *Scot. J. Geol.* **43**, 15–31 (2007).

12 Henry Cavendish, 'Experiments to determine the density of the Earth', *Phil. Trans. R. Soc. Lond.* **88**, 469–526 (1798).

13 David W. Hughes. 'The mean density of the Earth', *J. Brit. Astron. Assoc.* **116**, 21–24 (2006).

14 Charles Vernon Boys, 'On the Newtonian constant of gravitation'. *Phil. Trans. R. Soc. Lond. A* **186**, 1–72 (1895); Charles Vernon Boys, 'The Newtonian constant of gravitation', *Notices of the proceedings at the meetings of the members of the Royal Institution of Great Britain* **14**, 353–377 (1896). However, G has proved extremely difficult to measure precisely and even the current value, 6.67408×10^{-11} m^3 kg^{-1} s^{-2}, established in 2014 by the International Council for Science Committee on Data for Science and Technology (James E. Faller, 'Precision measurement, scientific personalities and error budgets: the *sine quibus non* for big G determinations', *Phil. Trans. R. Soc. Lond. A* **372**(2026) (2014), http://dx.doi.org/10.1098/rsta.2014.0023 (accessed 23 October 2019)) has an uncertainty of 0.00031×10^{-11} (National Institute of Standards and Technology, 'Fundamental physical constants. Newtonian constant of gravitation', *The NIST reference on constants, units, and uncertainty*, 2018, https://physics.nist.gov/cgi-bin/cuu/Value?bg (accessed 23 October 2019)).

15 Chandrasekhar, *op. cit.* (note 9), p. 373.

16 *Ibid*. See also John Henry Poynting, *The mean density of the Earth: an essay to which the Adams Prize was adjudged in 1893 in the University of Cambridge* (Charles Griffin, London, 1894); Keith Edward Bullen, *The Earth's density* (Chapman & Hall, London, 1975); and Hughes, *op. cit.* (note 13), for further discussion.

However, a second topic of interest had become apparent while Newton was preparing successive editions of the *Principia*: namely, the shape or 'figure' of the Earth. Between January and April 1691, observations of the planet Jupiter by the Italian astronomer Giovanni Cassini (1625–1712) had found its polar diameter to be shorter 'by a fifteenth part' than its equatorial diameter. This was later confirmed by observations by the English astronomer James Pound (1669–1724) in 1719 using a '123 foot telescope and an excellent micrometer',[17] who found it to be 1/13.[18] Then, in 1673, the French astronomer Jean Richer (1630–1696) found the length of a pendulum beating seconds at Cayenne (4.9° N, French Guiana) to be shorter than its previously measured length in Paris by a 'ligne ¼' (2.82 mm).[19] The notion of a planet's shape necessarily being that of a perfect sphere was no longer tenable.

These findings prompted Newton in the second (1713) edition of the *Principia*[20] to propose that a homogeneous, once fluid, Earth would achieve a state of hydrostatic equilibrium.[21] By considering the behaviour of two fluid-filled hypothetical 'canals' or 'columns' which extended from the centre to the surface of the body along the polar and equatorial semi-diameters, and which he assumed to balance or weigh the same, the whole attracting according to the universal inverse-square law, possessing a uniform density, and rotating about its axis of symmetry (i.e. the N–S polar axis), he showed that the body would take the form of an oblate ellipsoid whose ellipticity is infinitesimal. Regarding the Earth's mean density, he argued that:

> Now, I gather in the following way that the globe of our earth is denser than if it consisted totally of water. If this globe were wholly made of water, whatever things were rarer than water would because of their smaller specific gravity emerge from the water and float on the surface. … If the earth were not denser than the seas, it would emerge from those seas and, according to its degree of lightness, a part of the earth would stand out from the water, while all those seas flowed to the opposite side. By the same argument, the spots on the sun are lighter than the solar shining matter on top of which they float. And in whatever way the planets were formed, at the time when the mass was fluid, all heavier matter made for the center, away from the water. Accordingly, since the ordinary matter of our earth at its surface is about twice as heavy as water, and a little lower down, in mines, is found to be about three or four or even five times heaver than water, it is likely that the total amount of matter in the earth is about five or six times greater than it would be if the whole earth consisted of water. [22]

17 Newton 1729, *op. cit*. (note 8), vol. 2, p. 244.

18 *Ibid.* The mean of four sets of observations given in unnumbered table at p. 245: $(12.99 - 12.01)/12.99 \cong 1/13$. Its diameters are now known to be: polar 81 430 mi (131 049 km); equatorial 86 519 mi (139 239 km).

19 Anon., 'Année MDCLXXIV. Physique. Observations physiques', in *Histoire de l'Académie Royale des Sciences. Tome I. Depuis son établissement en 1666 jusqu'a 1686*, pp. 176–178 (I. G. Martin, J.-B. Coignard and H.-L. Guerin, Paris, 1733), at p. 7. A ligne was an archaic measure of length: 12 Paris lines = 1 Paris inch; 12 Paris inches = 1 Paris foot (32.48406 cm); 6 Paris feet = 1 Paris fathom (toise).

20 Andrew Motte's 1729 translation (note 8) was based on Newton's second (1713) and third (1726) Latin editions. See also John Martin Frederick Wright, *Commentary on Newton's Principia with a supplementary volume. Designed for the use of students at the universities*, vol. 1 (Black, Young and Young, London, 1828).

21 See Giuseppe Iurato, 'The dawning of the theory of equilibrium figures: a brief historical account from the 17th through the 20th century', in *XIII Congresso Società italiana di storia delle matematiche, Cagliari, 13–15 Nov. 2014*, 2014, https://arxiv.org/pdf/1409.3858.pdf (accessed 23 October 2019), for a history of the concept of an equilibrium figure.

22 Newton 1999, *op. cit*. (note 8), book 3, proposition 10, p. 815; Newton 1729, *op. cit.* (note 8), vol. 2, pp. 230–231.

Newton defined the degree of ellipticity (also known as *flattening*) as $f = \frac{d_E - d_P}{d_E}$, where d_E is the (mean) equatorial semi-diameter and d_P is the polar semi-diameter. Letting

$$\lambda = \frac{\text{centrifugal acceleration at the equator}}{\text{mean gravitational acceleration at the equator}},$$

he showed that, if g_E is the acceleration due to gravity at the equator and g_P is the acceleration due to gravity at the pole, then:

$$\frac{g_P}{g_E} = \frac{d_E}{d_P}(1 - \lambda) = \frac{1 - \lambda}{1 - f} \cong 1 + f - \lambda$$

and, for small f:

$$\frac{g_p}{g_E} = 1 + \frac{f}{5} = 1 + f - \lambda, \text{ and so } f = \frac{5}{4}\lambda^{23}.$$

Contemporary measurements had found that 'in the latitude of Paris [48° 50′ 10″] a heavy body falling in the time of one second, describes … 2,173$\frac{7}{9}$ lines' and

> a body revolving uniformly in a circle at a distance [R] of 19,615,800 [Paris] feet from the center, making a revolution in a single sidereal day of 23h56m4s, will describe an arc [ℓ] of 1,433.46 feet in the time of one second, an arc whose versed sine is [$\cong \ell^2/2R$] 0.0523656 feet, or 7.54064 lines. And therefore the force by which heavy bodies descend at the latitude of Paris is to the centrifugal force of bodies on the equator (which arises from the daily motion of the earth) as 2,174 to 7.54064.
>
> The centrifugal force of bodies on the earth's equator is to the centrifugal force with which bodies recede rectilinearly from the earth at the latitude of Paris … as the square of the radius to the square of the cosine of that latitude, that is, as 7.54064 to 3.267. Let this force be added to the force by which heavy bodies descend at the latitude of Paris; then a body falling at that latitude with the total force of gravity will, in the time of one second, describe 2,177.267 lines. … And the total force of gravity at that latitude will be to the centrifugal force of bodies on the earth's equator as 2,177.267 to 7.54064, or 289 to 1 [*i.e.* $\lambda = \frac{1}{289}$ and hence $f = \frac{1}{230}$].[24]

Unfortunately, because of the abstruse way in which Newton expressed his arguments, as well as omissions in his explanations, his theory 'struck even the most reputable continental mathematicians of his time as incomprehensible'.[25]

Both Chandrasekhar and Müller have reconstructed in some detail what Newton's chain of reasoning must have been.[26] Nevertheless, despite the difficulties met with by his contemporaries in understanding his arguments, his conclusion prompted many years

23 For discussion of the derivation of these equations, see Chandrasekhar, *op. cit.* (note 9), pp. 384–392; see also Wright, *op. cit.* (note 20), pp. 380–381.

24 Newton 1999, *op. cit.* (note 8), book 3, proposition 19, p. 822; Newton 1729, *op. cit.* (note 8), vol. 2, pp. 240–241. See also Chandrasekhar, *op. cit.* (note 9), pp. 390–392.

25 John L. Greenberg, *The problem of the Earth's shape from Newton to Clairaut: the rise of mathematical science in eighteenth-century Paris and the fall of 'normal' science* (Cambridge University Press, Cambridge, 1995), pp. 7 and 130.

26 Chandrasekhar, *op. cit.* (note 9), pp. 384–392; Wolfgang H. Müller, 'The effect of rotation on the flattening of celestial bodies: A journey through four centuries', *Math. Mech. Complex Syst.* **6**, 1–40 (2018) (dx.doi.org/10.2140/memocs.2018.6.1).

of practical investigations, conducted in many parts of the world by the most capable scientists and astronomers of the time, to measure both the length of a degree arc of latitude and the force of gravity at different latitudes, so as to determine the correct value of the figure of the Earth. Newton concluded, on the basis of the evidence of the few pendulum measurements available up to 1704, that 'the Earth is a little higher under the equator than by the … calculus, and a little denser at the centre than in mines near the surface'.[27]

Around 1730 the shape of the Earth became a reason for battle among the rival systems of British and Continental mechanics. Maupertuis (1698–1759), being exposed to Newton's ideas through his mentor, Johann Bernoulli, and equipped with the necessary mathematical skills, predicted that the Earth should be oblate,[28] while his scientific rival Jacques Cassini (1677–1756) claimed it to be prolate.[29] Variable-density models were subsequently considered by Legendre, Maclaurin and Laplace.[30] Remarkably, by assuming the Earth to be composed of a series of thin shells which gradually became more spherical as one approached the centre, and that its density (Δ) decreased outwards from the centre as a function of relative distance ($0 \le d \le 1$), $\Delta = \frac{\sin(md)}{d}$, where m is an empirical constant, Legendre determined that, with $\lambda = \frac{1}{288}$ and setting $m = \frac{7\pi}{8}$, the ellipticity at the surface would be $\frac{1}{319}$ and at the centre $\frac{1}{424}$, giving 'a flattening of the Earth conforming to that which we adopt from pendulum measurements' and a density at the centre approximately seven times that at the surface.[31]

By the 1870s, numerous publications discussing mathematical theories of attraction and the figure of the Earth had appeared, and determinations of flattening based on gravimetric measurements were converging on what is now the accepted value (first established in the late 1950s using Earth-orbiting satellites and refined subsequently) of 1/298.24 (figure 1).[32] Pratt's book on the figure of the Earth (with which Adcock was familiar—see below) confirmed that, by assuming that the ellipticity of the strata decreased and their density increased as one approached the centre of the Earth, and taking a mean density for the Earth of 6.66 g cm^{-3} (based on the result obtained by George Biddell Airy (1801–1892) from experiments at the Harton Pit coalmine in Derbyshire, England), the estimated

27 Newton 1729, *op. cit.* (note 8), p. 250.

28 Pierre Louis Moreau de Maupertuis, 'Article V: comment les expériences sur la pesanteur pouvoient faire croire que la Terre n'étoit pas sphérique', in *Éléments de géographie*, pp. 28-34 (Gab. Martin, J. Bapt. Coignard and Hipp. L. Guerin, Paris, 1742), at pp. 31–42.

29 Jacques Cassini, *Traité de la grandeur et de la figure de la terre* (Pierre de Coup, Amsterdam, 1723).

30 Adrien-Marie Legendre, 'Suite des recherches sur la figure des planètes', in *Histoire de l'Académie des Sciences. Année MDCCLXXXIX. Avec les Mémoirés de Mathématique et de Physique, pour la même année. Tirés des registres de cette Académie*, at pp. 372–454 ([Paris], 1793); Colin Maclaurin, *A Treatise on Fluxions. In two volumes. Second edition. To which is prefixed an account of his life* (William Baynes and William Davis, London, 1801); Pierre-Simon Laplace, *Traité de mécanique céleste. Tome cinquième* (Bachelier, Paris, 1823). See also Greenberg, *op. cit.* (note 25); Isaac Todhunter, *A history of the mathematical theories of attraction and the figure of the earth: from the time of Newton to that of Laplace*, 2 vols in 1 (Macmillan, London, 1873), for discussion.

31 Legendre, *op. cit.* (note 30), p. 411. The currently accepted value for the mean density of the whole Earth is 5.52 g cm^{-3}; and that of the core, which occupies the innermost 55%, is 16.9 g cm^{-3}. See Adam M. Dziewonski and Don L. Anderson, 'Preliminary reference Earth model', *Phys. Earth Planet. In*. **25**, 297–356 (1981).

32 Data from Newton 1729, *op. cit.* (note 8); Todhunter, *op. cit.* (note 30); Georg Straßer, 'Ellipsoidische Parameter der Erdfigur (1800–1950)', in *Deutsche Geodätische Kommission bei der Bayerischen Akademie der Wissenschaften. Reihe A. Höhere Geodäsie, 19* (Bayerischen Akademie der Wissenschaften, Munich, 1957); and David Richard Williams, 'Planetary fact sheet: metric', 2018, https://nssdc.gsfc.nasa.gov/planetary/factsheet (accessed 23 October 2019).

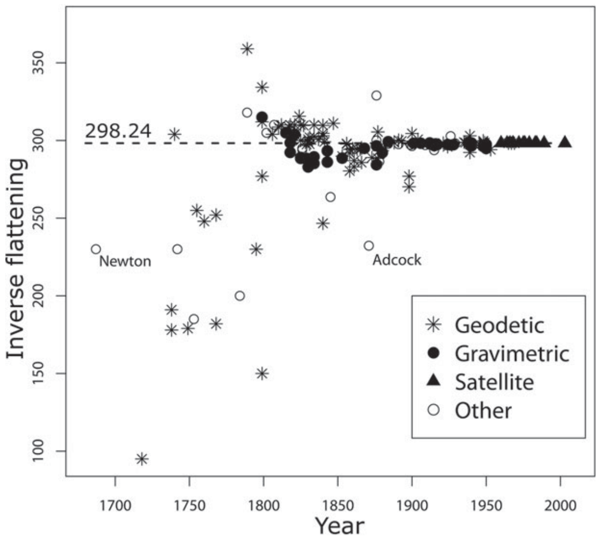

Figure 1. Determinations of the flattening or 'figure' of the Earth using geodetic, gravimetric (pendulum) and Earth-orbiting satellite measurements ('other' includes purely theoretical calculations) since the time of Newton, expressed as inverse flattening values for convenience. The dashed line shows the currently accepted flattening value of 1/298.24.

flattening was $\frac{1}{307}$.[33] Pratt concluded that the discrepancy between Newton's estimate of an inverse flattening of 230 and one of approximately 300 could therefore be explained by having an inhomogeneous Earth whose density increased from the surface to the centre, rather than a homogeneous model.[34]

On 26 August 1872, Adcock gave a paper at the 21st meeting of the American Association for the Advancement of Science at Dubuque, Iowa, titled 'The force at any point of the surface of a rotating fluid ellipsoid of three unequal axes, under the action of the gravity of its own particles and the accompanying centrifugal force'. He seems to have regarded this work as being of major importance, as he had already gone to the expense of paying for a written version, with the title 'Gravitation to the sphere and the two ellipsoids of revolution: ratio of the axes of a rotating fluid mass', to be printed (see below), and he had sent a copy of it to his local newspaper, the *Monmouth Review* (Monmouth, Illinois), in the last week of January 1872, as well as having announced it in a monthly church newspaper, the *Gospel Echo and Christian* (Quincy, Illinois).[35] Thereafter, for several months, he advertised it for sale at a price of $2 (equivalent to about $40 today[36]) in the *Journal of the Franklin Institute* and the *American Journal of Science and Arts*. He also

33 John Henry Pratt, *A treatise on attractions, Laplace's functions, and the figure of the Earth* (Macmillan, London, 1860), p. 79. George Biddell Airy, 'Account of pendulum experiments undertaken in the Harton Colliery, for the purpose of determining the mean density of the earth. & … Supplement to the "Account of pendulum experiments undertaken in the Harton Colliery"; being an account of experiments undertaken to determine the correction for the temperature of the pendulum', *Phil. Trans. R. Soc. Lond.* **146**, 297–355 (1856).

34 Pratt, *op. cit.* (note 33), p. 65.

35 *Chicago Tribune*, 30 March 1872, p. 4 (https://chroniclingamerica.loc.gov/search/titles/).

36 Based on the Consumer Price Index (https://www.measuringworth.com/calculators/uscompare/relativevalue.php).

sent copies to the Library of Congress, Washington; Transylvania University, Lexington, Kentucky; Yale University, New Haven, Connecticut; the Washburn Observatory of the University of Wisconsin, Madison, Wisconsin; and the Royal Society in London.

'Gravitation to the sphere and the two ellipsoids of revolution: ratio of the axes of a rotating fluid mass' by R. J. Adcock

The text which follows is a transcription of the quarto-sized pamphlet which Adcock presented to the Royal Society.[37] Because of the wider page size, the symbol | is used here to denote the end of each line in the original. His paragraph insets, type style, etc. are retained. His final printed page consists of a list of 39 corrections of typographical errata; for ease of reference, these are given here as footnotes with the attribution 'RJA'. In addition, we note several errors which he appears to have missed. A commentary on his text follows the transcript.

Front cover (page 1)

GRAVITATION| TO THE| SPHERE AND THE TWO ELLIPSOIDS OF REVOLUTION:| RATIO OF THE AXES| OF A| ROTATING FLUID MASS.|
BY R.J. ADCOCK.| CINCINNATI, O.:| WILSTACH, BALDWIN & Co., Printers.| 1872.

Inside front cover (page 2)

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(Bound-in insert following page 2)

CIRCULAR.|

ATTRACTION OF THE SPHERE AND THE TWO| ELLIPSOIDS OF REVOLUTION; Ratio of the Axes of a Rotating| Fluid Mass under the action of the gravity of its own| Particles. By R.J. ADCOCK.| Printed by Wilstach, Bald-|win & Co.,| Cincinnati, Ohio.

For a Post Office Order of Two Dollars, drawn on or pay-|able at the Post Office at Galesburg,[38] *I propose to send| by mail the above work of eight pages quarto, suitable generally| for Professors of Mathematics and advanced Students, in which is| published for the first time, a correct solution of the latter prob-|lem which has employed so much thought and expense, by indi-|viduals since the time of Newton.*

Address,| R.J. ADCOCK,| Utah,[38] Warren Co., Illinois.|

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ATTRACTION OF A CONICAL VOLUME.|

The attraction of matter concentrated to a point, being directly as its mass and inversely as the square of its| distance from the attracted point. Let|

37 Tracts 342, Royal Society Library, London.

38 In the copy in the Royal Society Library, both 'Galesburg' and 'Utah' are crossed through in pencil and changed to 'Monmouth'. The same text also appears as an advertisement in *J. Franklin Institute*, Philadelphia, **113** (1872), pp. 216, 362 and xxiii, but with 'Monmouth' substituted.

39 Unnumbered page in the original.

$k_{,}$ = the attraction of a unit mass concentrated at the distance of unity from the attracted point.|

δ = the density of the attracting material.|

$d\theta$ and $r^2 d\theta$ = the infinitely small surfaces spherical at the distances unity and r from the center[40] of a sphere.|

Then the attraction of the elementary volume $r^2 d\theta dr$ for the center of the sphere is|

$$\frac{\delta k_{,} r^2 d\theta dr}{r^2} = \delta k_{,} d\theta dr, \text{ (1.)|}$$

which is independent of r, the distance of the attracting material from the attracted point.|

INTEGRATING EQUATION (1.)|

$\int_0^{\theta} \int_{r=r_2}^{r=r_1} \delta k_{,} d\theta dr = \delta' k_{,} \theta (r_1 - r_2)$, (2.) which is the attraction of a conical or pyramidal frustrum for a| point at its vertex, and is independent of its distance from the vertex.|

ATTRACTION OF A SPHERE AND SPHERICAL SHELL.|

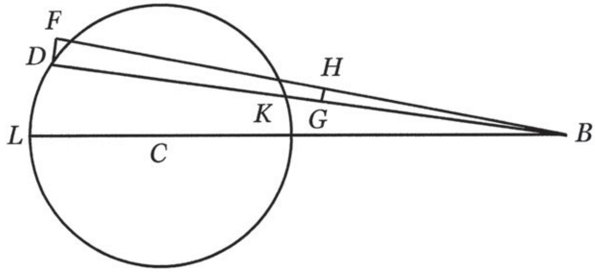

Figure 2. This figure had no caption in the original manuscript and was placed in the upper right of the corresponding page.

Let θ = the arc at a unit's distance from the center, and the| measure of the angle it subtends at the center of the arc = the| angle CBD [figure 2].|

$d\theta$ = the angle DBF.|

du = the elementary arc of rotation of the plane CBD| about BC as an axis. Then|

$d\theta du$ sineθ measures the solid angle at B, the vertex of| the pyramidal frustrum, described by DGHF, through the angle| du about BC. Let|

DG = Ce = the distance through the attracting material.|

By formula (2.) $\delta k_{,} Ce$ sine θ cos.θ $d\theta du$ = the component| in the direction BC of the attraction of the frustrum for the| point B. The double integral of which between the proper| limits, $\int_0^n \int_0^{\theta} \delta k_{,}$sine θ cos.θ $d\theta du$ [41] × Ce (3.)[42] is the general expression for the attraction of any geometrical volume| of homogeneous density.|

40 RJA: for center read centre, and at all other places where center occurs.

41 The n in the upper bound of the first integral is most likely confused with the letter u.

42 RJA: after Ce and before (3.), insert a comma.

For the sphere, let $CL = r$, $CB = a > r$. Then $DK = Ce = (r^2 - a^2\text{sine }\theta^2)^{1/2}$, and (3.) becomes|

$\int_0^{u=2\pi} \int_0^{\theta=\text{sine}'\frac{r}{a}} 2\delta k_{\prime}\ (r^2 - a^2\text{sine}^2\theta)^{1/2}\text{sine }\theta\cos.\theta d\theta du =$

$4\pi\delta k_{\prime} \int_0^{\theta=\text{sine}'\frac{r}{a}} (r^2 - a^2\text{sine}^2\theta)^{1/2}\text{sine }\theta|\cos.\theta da = \frac{4\pi\delta k'}{3a^2}$[43]$(r^2 - a^2\text{sine}^2\theta)^{\frac{3}{2}} + C = \frac{4\pi\delta k_{\prime} r^3}{3a^2}$, the same as its mass $\frac{4}{3}\pi r^3\delta$, concentrated in its center.|

Differentiating, $d\ \frac{4}{3}\ \frac{\pi r^3\delta k_{\prime}}{a^2} = \frac{4\pi\delta k_{\prime} r^2 dr}{a^2} =$ the attraction of the spherical shell for the exterior point B.|

And by (1.) the attraction of the shell for an interior point is zero, the attractions of the opposite elementary| frustrums being equal and opposite.

PARTICULAR CASE OF THE ATTRACTION OF THE PROLATE ELLIPSOID| OF REVOLUTION.|

The distance between the points in which the line $y' - y = \tan.\theta(x - x')$ intersects the ellipse $a^2y^2 + b^2x^2 = a^2b^2$ is|

$$Ce = \frac{2ab(a^2\tan.^2\theta + b^2 - 2x'y'\tan.\ \theta - y'^2 - \tan.^2\theta x'^2)^{1/2}(1 + \tan.^2\theta)^{1/2}}{a^2\tan.^2\theta + b^2}, \ (4.)|$$

Hence, for the prolate ellipsoid of revolution and exterior point, (x', y'), on longer axis at the distance h from the| center, where $y' = 0$ and $x' = h$, formula (3.) gives| $\int 2\pi\delta k_{\prime} Ce \text{ sine }\theta\cos.\theta\, d\theta = \int \frac{4\ \pi ab\ (a^2\tan^2\theta + b^2\cos.^2\theta - h^2\text{sine}^2\theta)^{1/2}\ \text{sine }\theta\cos.\ \theta\, d\theta}{a^2\text{sine}^2\theta + b^2\cos.^2\theta}$ (5.)|[44]

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in which, for the whole ellipsoid, the limits are $\theta = 0$, $\theta = \tan.^{-1}\frac{b}{(h^2-a^2)^{1/2}}$. Let $(a^2 - h^2)$ $\text{sine}^2\theta + b^2\cos.^2\theta| = z^2$, then when $\theta = 0, z = b$, and when $\theta = \tan.^{-1}\frac{b}{(h^2-a^2)^2}$,[45] $z = 0$, and the integral of (5.) is|

$$-\ \delta k_{\prime} \int \frac{4\pi abz^2dz}{b^2h^2 - c^2z^2} = \frac{4\pi a^2b\delta k_{\prime}}{c^2}\left(z - \frac{1}{2}\frac{bh}{c}\log.\frac{bh + cz}{bh - cz}\right) + C, \quad (6.)|$$[46]

43 The integration should be with respect to $d\theta$ not da. There is a minus sign missing after the integration is carried out. However, this does not matter if the upper and lower bounds are inserted. The final result after the integration is correct.

44 RJA: eq. (5.), for $\tan.^2\theta$ read $\text{sine}^2\theta$.

45 RJA: for $\tan.^{-1}\frac{b}{h^2-a^2}$ read $\tan.^{-1}\frac{b}{(h^2-a^2)^{1/2}}$.

46 RJA: for a^2b read ab^2; note that where Adcock uses the symbol 'log.', he actually means the natural logarithm, i.e. 'ln'.

$= \int_{z=0}^{z=b} 2\pi C_e \delta k_\prime$ sine $\theta \cos.\theta\, d\theta = \frac{4\pi ab^2 \delta k_\prime}{c^2}\left(1 - \frac{h}{2c}\log.\frac{h+c}{h-c}\right.$ (7.),[47] which is the attraction of the prolate ellip-|soid of revolution for an exterior point on the major axis, at a distance h from the center.|

Differentiating (7.) with respect c,[48] $\frac{a}{b}$ and h and θ being constant|

$$\delta k_\prime \int_{\theta=\tan.^{-1}\frac{b}{(h^2-a^2)^{1/2}}}^{\theta=0} 2\pi D c C_e dc \text{ sine } \theta \cos.\theta d\theta$$
$$= Dc\frac{4\pi a(a^2-c^2)\delta k_\prime}{c^2}\left(1 - \frac{h}{2c}\log.\frac{h+c}{h-c}\right)dc = \frac{4\pi ab^2\delta k_\prime dc}{c(h^2-c^2)}, \text{ (8.)|}$$

which is the attraction of the prolate ellipsoidal shell, whose outer and inner surfaces are similar concentric and simi-||larly placed, for a point at the distance h from the center on the major axis.|

A PARTICULAR CASE OF THE OBLATE ELLIPSOID OF REVOLUTION.|

For an exterior point on the minor axis of the oblate ellipsoid of revolution, in (4.) for θ, $\pi - \theta$ [49] must be| substituted, and formula (3.) becomes|

$$\int_{\theta=0}^{\theta=\tan.^{-1}\frac{a}{(h^2-b^2)^{1/2}}} 2\pi ab\delta k_\prime C_e \text{ sine } \theta \cos.\theta d\theta$$
$$= \int_{\theta=0}^{\theta=\tan.^{-1}\frac{a}{(h^2-b^2)^{1/2}}} \frac{4\pi ab\delta k_\prime (a^2\cos.^2\theta - (h^2-b^2)\text{sine}^2\theta)^{1/2}}{a^2\cos.^2\theta + b^2\text{sine}^2\theta} \times |\text{sine } \theta \cos.\theta d\theta. \text{ (9.)|}$$

Let $a^2\cos.^2\theta - (h^2-b^2)\text{sine}^2\theta = z^2$, then the preceding equation becomes|

$$= \int_{z=a}^{z=0} \frac{4\pi ab\delta k_\prime z^2 dz}{a^2h^2 + c^2z^2} = -\frac{4\pi a^2 b\delta k_\prime}{c^2}\left(z - \frac{ah}{c}\tan.^{-1}\frac{cz}{ah}\right) + C$$
$$= \frac{4\pi a^2 b\delta k_\prime}{c^2}\left(1 - \frac{h}{c}\tan.^{-1}\frac{c}{h}\right), \text{ (10.)}$$

which is the attrac-|tion of the oblate ellipsoid of revolution for an exterior point on its minor axis, at the distance h from the centre.|

47 RJA: insert before (7.),).
48 'with respect c' should read 'with respect to c'.
49 RJA: for π read $\frac{1}{2}\pi$.

Differentiating (10.) with respect to c, considering $\frac{b}{a}$, $\frac{c}{a}$, θ, and h constant, there results|

$$\delta k_{\prime} \times \int_{\theta=0}^{\theta=\tan.^{-1}\frac{a}{(h^2-b^2)^{1/2}}} 2\pi \text{ sine } \theta \cos.\theta d\theta \times DcC_e dc = Dc\frac{4\pi a^2 b\delta k_{\prime}}{c^2}\left(1-\frac{h}{c}\tan.^{-1}\frac{c}{h}\right)dc$$

$$= \frac{4\pi a^2 b\delta k_{\prime} dc}{c(h^2+c^2)}, \quad (12.)|$$

which is the attraction of the oblate ellipsoidal infinitely thin shell, whose outer and inner surfaces are similar,| concentric and similarly placed, for a point on the minor axis at a distance h from the center.|

$\frac{4\pi a^2 bdc}{c}$ is the volume of the shell. Ellipsoidal shells of any thickness having their surfaces so situated, have| their attractions for a point interior to the inner surface zero. For if through the point two opposite infinitessimal[50]| cones be drawn, the distance through the material in the opposite directions are equal, being the semi differences of two| cords, which are the projections of two cords of the two circles of which the elliptic sections are the projections.| And, therefore, by (1.), their attractions are equal.

DIRECTION OF THE ATTRACTION OF THE ELLIPSOIDAL SHELL.|

The cord to the ellipse, $C_e = \frac{2ab(a^2\alpha^2+b^2+2x'y'-y'^2-x'^2\alpha^2)^{1/2}(1+\alpha^2)^{1/2}}{a^2\alpha^2+b^2}$, (13)[51] in which $\alpha = \tan.\frac{a}{C_e}$ = the| tangent of the angle which C_e makes with a.|

Differentiating C_e with respect to c; $\frac{a}{b}$,α,x',y', being constant $dC_e = \frac{2ab(1+\alpha^2)^{1/2}dc}{c(a^2\alpha^2+b^2+2x'y'-y'^2-x'^2\alpha^2)^{1/2}}$.[52] (14.)|

Put $\alpha = \frac{\beta\mp\delta}{1\pm\delta\beta}$, $\beta = \tan.\frac{b_{\prime}}{a} = \frac{\alpha_1^2 y'}{(\alpha_{\prime}^2-c^2)x}$, $\delta = \tan.\frac{b_{\prime}}{C_e}$, $b_{\prime}$ = the line from (x',y') to the axis of x, bisecting the| angle between the two lines from (x',y') to the foci.|

$a_{\prime}$ = semi major axis of the ellipse passing through the point (x', y'), and having the same foci as that whose| semi axis is a. Then|

$$dC_e = \frac{2abdc}{c}\left(\frac{1+\frac{\beta\mp\delta}{1\pm\beta\delta}}{\alpha^2\left(\frac{\beta\mp\delta}{1\pm\beta\delta}\right)^2+b^2+2x'y'\frac{\beta\mp\delta}{1\pm\beta\delta}-y'^2-x'^2\left(\frac{\beta\mp\delta}{1\pm\beta\delta}\right)^2}\right)^{\frac{1}{2}} = |^{53}$$

$$\frac{2abdc}{c}\left(\frac{(1+\beta^2)(1+\delta^2)}{a^2\beta^2+a^2\delta^2+b^2+b^2\delta^2\beta^2-2x'y'\beta\delta^2-y'^2-y'^2\beta^2\delta^2-\beta^2x'^2-x'^2\delta^2}\right)^{\frac{1}{2}}, (15.)|$$

50 'infinitessimal' should read 'infinitesimal'.
51 RJA: for $2x'y'$ read $2\alpha xy'$.
52 RJA: for $2x'y'$ read $2\alpha xy'$.
53 RJA: for $1+\frac{\beta\mp\delta}{1\pm\beta\delta}$ read $1+\left(\frac{\beta\mp\delta}{1\pm\beta\delta}\right)^2$; for α^2 read a^2.

from which the first power of s[54] having disappeared, it follows that the differentials of cords on opposite sides of $b_{\prime}$| making equal angles with it are equal, and, therefore, by (2.) the elementary attractions $\delta k_{\prime} dC_e$sine $\theta \cos.\theta d\theta$,|

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$\delta k_{\prime} dC_e$sine$(-\theta)$ cos.$(-\theta)$ $d(-\theta)$ are equal, and the same being true for every section through $b_{\prime}$, it must be the| direction of the attraction. Therefore, the direction of the attraction of ellipsoidal shells whose outer and inner sur-|faces are similar, concentric, and similarly placed, for an exterior point, is that of the line bisecting the angle| between the lines drawn to the foci on the longest axis of the ellipsoid. And is, therefore, perpendicular to the sur-| face of the ellipsoid passing through the given point (x',y') and having the same foci as that of the outer surface of| the shell.

ATTRACTION OF SHELLS.|

Let dN = the infinitely small thickness of a plane lamina. Then $\frac{dN}{\cos.\theta} = C_e$ = the distance through it in any| direction θ. And by (3.) $\int_{\frac{1}{2}\pi}^{0} 2\pi\delta k_{\prime} \frac{dN}{\cos.\theta}$sine θ cos. $\theta d\theta = 2\pi\delta k_{\prime} dN + C = 2\pi\delta k_{\prime} dN$, (16.) = the attraction of| an infinitely thin lamina, infinitely extended, for a point at any distance, = the attraction at distance zero.|

By a formula on page 2,[55] $\frac{4\pi\delta k_{\prime} r^2 dr}{a^2}$ being the attraction of a spherical shell at the distance a from the center,| $4\pi\delta k_{\prime} dr$ is the attraction for a point on the exterior surface of the shell, and is normal to the surface. The two| parts of a cord between the inner and outer surfaces being equal, $2\pi\delta k_{\prime} dr$ is the attraction of all the parts of the| shell at the first or second intersection. Therefore, the attraction of the part at the first intersection of the cord with| any double curved lamina, whose two surfaces have a common normal at the point, is $2\pi\delta k_{\prime} dr$, because it is evidently| between those of the two spherical lamina of greatest and least curvature, each of which is $2\pi\delta k_{\prime} dr$, dr = the thick-|ness. Hence the attraction, for a point on the exterior surface of any closed shell, of an infinitessimal[56] thickness,| which has the parts of any cord, passing through the given point, between the inner and outer surfaces of the shell,| equal, is $4\pi\delta k_{\prime} dN$. That is, A being the attraction of the shell $\frac{A}{dN} = 4\pi\delta k_{\prime}$, (17.) at the limit.|

DIFFERENTIATING UNDER THE SIGN OF INTEGRATION.|

By the proposition that integrals and differentials are relatively free, it follows that the integral of the differential| coefficient of the quantity under the sign of integration, with respect to any quantity, equals the differential coefficient| of the integral with respect to the same quantity. Giving when $\int du = \omega$, whether general or definite,|

$\int D_h du = D_h \omega = \frac{d\omega}{dh}$, (18.)|

54 RJA: for 'the first power of s' read 'odd powers of δ'.
55 RJA: for 2 read 3.
56 'infinitessimal' should read 'infinitesimal'.

POTENTIAL.|

Let the term potential = ω, be the general integral of the elementary quantity of work of forces directed to and| dependent on the distances to fixed centers from the attracted point.|

A level surface is that which has the potential at every point constant.|

And, consequently, perpendicular to the resultant of the forces at every point, for the elementary quantity of| work, which is the differential of the potential, is zero in no other direction.|

R = resultant of forces, and normal to the level surface.|

$dN_{\prime}$ = the distance perpendicular to both, between two level surfaces, infinitely close.|

$dl = \frac{dN_{\prime}}{\cos.\frac{N_{\prime}}{l}}$ = the distance between them in the direction l. Then $\frac{d\omega}{dN_{\prime}} = R$ (19.), and $\frac{d\omega}{dl} =$ the component of| R in the direction l.

ATTRACTION OF THE CONCENTRATED ELLIPSOID.|

Let a, b, $b_{\prime}$, be the semi axes of the ellipsoid, a the greatest and $b_{\prime}$ the least.|

u = the angle which a plane through $b_{\prime}$ makes with a.|

a' = the major axis of the section. And $a'^2 - b_{\prime}^2 = c_{\prime}^2$, (20.)|

Then $a^2a'^2\text{sine}^2u + b^2a'^2\cos.^2u = a^2b^2$, $a'^2 = \frac{a^2b^2}{a^2 - c^2\cos.^2u}$, (21.) By (18.) and (12.), $\dfrac{2\delta k_{\prime}dua'^2b_{\prime}dc_{\prime}}{c_{\prime}(c'^2+h^2)}$ =| the attraction of a portion of an ellipsoidal shell for a point on its shorter axis at the distance h from the center,| comprehended between two planes through that axis making the angle du with each other, and the angle u with| the longest axis.|

Since the outer and inner surfaces are similar, $dc_{\prime} = \dfrac{c_{\prime}dc}{c}$. By (20.), (21.) $c_{\prime}^2 = \dfrac{a^2b^2 - a^2b_{\prime}^2 + b^2c^2\cos.^2u}{a^2 - c^2\cos.^2u}$.|[57]

$$\text{Then } 2\int_{u=0}^{u=\frac{1}{2}\pi}\frac{2dua'^2b_{\prime}dc_{\prime}\delta k_{\prime}}{c_{\prime}(c_{\prime}^2+h^2)} = 2\int_{u=0}^{u=\frac{1}{2}\pi}\frac{2dua^2b^2b_{\prime}dc\delta k_{\prime}}{c(a^2b^2 - a^2b_{\prime}^2 + b_{\prime}^2c^2\cos.^2u + a^2h^2 - c^2h^2\cos.^2u)} = |$$

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$\frac{4\pi\delta k_{\prime}abb_{\prime}dc}{c(b^2-b_{\prime}^2+h^2)^{1/2}(a^2-b_{\prime}^2+h^2)^{1/2}}$, in which c, $(a^2 - b_{\prime}^2)^{\frac{1}{2}}$, $(b^2 - b_{\prime}^2)^{\frac{1}{2}}$ are the semi focal distances of the ellip-|soid, and $\frac{4\pi abb_{\prime}dc}{c}$ is the volume of the shell. Let ω = the potential of the shell, then by (19.) with respect to h,| $\int\frac{4\pi abb_{\prime}\delta k_{\prime}dcdh}{c(b^2-b_{\prime}^2+h^2)^{1/2}(a^2-b_{\prime}^2+h^2)^{1/2}} = \omega$, from which it is evident that shells, of the same foci, whose outer and| inner surfaces are similar and similarly placed

57 RJA: for $b^2c^2\cos.^2u$ read $b_{\prime}^2c^2\cos.^2u$.

concentric ellipsoidal surfaces, have their potentials proportional to| their masses, and, therefore, their attractions for any exterior point in the same ratio, since by proposition, page| 6,[58] and equation (19.), the level surface of the shell is an ellipsoidal surface of the same foci. Let these two ellip-|soids of the same foci be divided into the same number of similar shells, the masses of the shells will be as the| masses of the ellipsoids of which they are parts, the component in any direction of the attraction of a shell of one will| be to the component in the same direction of the attraction of the corresponding shell of the other ellipsoid as their| masses, as the masses of the ellipsoids, and, therefore, the sums of these components, or the attractions of the ellipsoids| themselves in any direction, are as their masses. Hence, the attraction of the ellipsoid of homogeneous density| equals the attraction of its mass concentrated into the ellipse whose semi axes are the focal distances $(a^2-b_{\prime}^2)^{1/2}$.| $(a^2-b_{\prime}^2)^{1/2}$. The density which compressed mass at any point is $\frac{abb_{\prime}\delta}{c^3}$ multiplied by the double ordinate of the| ellipsoid of revolution whose major and minor axes are $2(a^2-b_{\prime}^2)^{1/2}$, $2(b^2-b_{\prime}^2)^{1/2}$.|

ATTRACTION OF THE PROLATE ELLIPSOID OF REVOLUTION.

Since, as just proved, the attractions of ellipsoids of the same foci are as their masses, the attraction of the| prolate ellipsoid of revolution, for an exterior point, equals the attraction of the ellipsoid, of equal mass, whose semi-|axes are the focal distance c and zero. The quantity of matter concentrated on $2c$ at any point being proportional to| c^2-x^2, x being the distance from the center, and the original density δ being increased by the ratio of ab^2 to c. So| that $\frac{ab^2\delta}{c}$ being the density at the center, $c^2 : c^2-x^2 :: \frac{ab^2\delta}{c} : \frac{ab^2\delta(c^2-x^2)}{c^3}$ = the density at the point x. Hence,| the component, perpendicular to the axis of x, of the attraction of the prolate ellipsoid of revolution for an exte-|rior point (x',y') is

$$\frac{\pi ab^2\delta k_{\prime}}{c^3}\int_{x=-c}^{x=c}\frac{(c^2-x^2)y'dx}{(y'^2+(x'-x)^2)^{\frac{3}{2}}} = \delta k_{\prime}\Bigg(-\frac{\pi ab^2 y'}{c^3} \times \int\left(\frac{(c^2-x'^2)dz}{(y'^2+z^2)^{\frac{3}{2}}}+\frac{2x'zdz}{(y'^2+z^2)^{\frac{3}{2}}}-\frac{z^2dz}{(y'^2+z^2)^{\frac{3}{2}}}\right)+C^{59}|$$

$$=-\frac{\pi ab^2\delta k_{\prime}y'}{c^3}\left(\frac{(c^2-x'^2)z}{y'^2(y'^2+z^2)^{1/2}}-\frac{2x'}{(y'^2+z^2)^{1/2}}+\frac{z}{(y'^2+z^2)^{1/2}}-\frac{1}{2}\log.\frac{(y'^2+z^2)^{1/2}+z}{(y'^2+z^2)^{1/2}-z}\right)+C=|$$

$$\frac{\pi ab^2\delta k_{\prime}y'}{c^3}\left(-\frac{(c^2-x'^2)(x'-x)}{y'^2(y'^2+(x'-x)^2)^{1/2}}+\frac{x'-x}{(y'^2+(x'-x)^2)^{1/2}}-\frac{1}{2}\log.\frac{(y'^2+(x'-x)^2)^{1/2}+x'-x}{(y'^2+(x'-x)^2)^{1/2}-x'+x}\right)+C|$$

58 RJA: for 6 read 4; and for 'these' read 'then'.

59 for ') + C' read ')) + C.'

$$=\frac{\pi ab^2\delta k_{\prime}y'}{c^3}\left\{\begin{array}{l}\dfrac{(x'^2-c^2)(x'-c)}{y'^2(y'^2+(x'-c)^2)^{1/2}}+\dfrac{x'+c}{(y'^2+(x'-c)^2)^{1/2}}+\dfrac{1}{2}\log.\dfrac{(y'^2+(x'-c)^2)^{1/2}+x'-c}{(y'^2+x'-c)^2-x'+c}\\ -\dfrac{(x'^2-c^2)(x'+c)}{y'^2(y'^2+(x'+c)^2)^{1/2}}-\dfrac{x'-c}{(y'^2+(x'+c)^2)^{1/2}}-\dfrac{1}{2}\log.\dfrac{(y'^2+(x'+c)^2)^{1/2}+x'+c}{(y'^2+(x'+c)^2)^{1/2}-x'-c}\end{array}\right\}|$$[60]

Let $y'^2=\frac{b_{\prime}^2}{a_{\prime}^2}(a_{\prime}^2-x'^2)=\left(a_{\prime}\pm\frac{cx'}{a_{\prime}}\right)^2-(x'\pm c)^2=a_{\prime}^2+\frac{c^2x'^2}{a_{\prime}^2}-x'^2-c^2$, *then*|

$(y'^2+(x'+c)^2)^{1/2}=a_{\prime}+\frac{cx'}{a_{\prime}}$, and $(y'^2+(x'-c)^2)^{1/2}=a_{\prime}-\frac{cx'}{a_{\prime}}$ and|

$$\frac{1}{2}\log.\left(\frac{(y'^2+(x'+c)^2)^{1/2}+x'+c}{(y'^2+(x'+c)^2)^{1/2}-x'-c}\times\frac{(y'^2+(x'-c)^2)^{1/2}-x'+c}{(y'^2+(x'-c)^2)^{1/2}+x'-c}\right)$$

$$=\frac{1}{2}\log.\left(\frac{a_{\prime}+\frac{cx'}{a_{\prime}}+x'+c}{a_{\prime}+\frac{cx'}{a_{\prime}}-x'-c}\times\frac{a_{\prime}-\frac{cx'}{a_{\prime}}-x'+c}{a_{\prime}-\frac{cx'}{a_{\prime}}+x'-c}\right)|$$

$$=\frac{1}{2}\log.\frac{(a_{\prime}+c)^2-(a_{\prime}+c)^2\frac{x'^2}{a_{\prime}^2}}{(a_{\prime}-c)^2-(a_{\prime}-c)^2\frac{x'^2}{a_{\prime}^2}}=\log.\frac{a_{\prime}+c}{a_{\prime}-c}=2\log.\frac{c+a_{\prime}-b_{\prime}}{c-a_{\prime}+b_{\prime}},\text{ so that}|$$

$$\int_{-c}^{c}\frac{\pi ab^2\delta k_{\prime}y'(c^2-x^2)dx}{c^3(y'^2+(x'-x)^2)^{3/2}}=\pi ab^2\delta k_{\prime}y'\left(\frac{2a_{\prime}c}{b_{\prime}^2}-\log.\frac{a_{\prime}+c}{a_{\prime}-c}\right)=\text{the component}$$

perpendicular to the major axis, of| the attraction of the prolate ellipsoid of revolution for an exterior point (x',y'), $a_{\prime}$ and $b_{\prime}$ being the axes of the level| surface passing through it.|

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GRAVITATION| [running head]

The component parallel to the same axis|

$$=\int\frac{d\omega'}{dN_e}\times\frac{a_{\prime}^2b_{\prime}^2x'}{a_{\prime}^2(a_{\prime}^4y'^2+b_{\prime}^4x'^2)^{1/2}}=\int_0^c\frac{4\pi ab^2b_{\prime}^2x'dc\delta k_{\prime}}{a_{\prime}c(c^2y'^2+b_{\prime}^4)}=\frac{4\pi\delta k_{\prime}ab^2x'}{c^3}\int_{a_{\prime}}^{(x'^2+y'^2)^{1/2}}\frac{b_{\prime}^2cda_{\prime}}{a_{\prime}^2y'^2}|$$

60 RJA: the sign before *c*, in the numerators of the 2nd and 5th terms, to be changed.

$$= \frac{4\pi ab^2\delta k_\prime x'}{c^3}\int_{a_\prime}^{(x'^2+y'^2)^{1/2}} \frac{(a_\prime^2 - x'^2 - y'^2)^{1/2} a_\prime da_\prime}{(a_\prime^2 - x^2)^{\frac{3}{2}}} = \frac{4\pi ab^2\delta k_\prime x'}{c^3}\left(\frac{(a_\prime^2 - x'^2 - y'^2)^{1/2}}{(a_\prime - x'^2)^{1/2}}\right) - |$$

$$\frac{1}{2}\log.\frac{(a_\prime^2 - x'^2)^{1/2} + (a_\prime^2 - x'^2 - y'^2)^{1/2}}{(a_\prime^2 - x'^2)^{1/2} + (a_\prime^2 - x'^2 - y'^2)^{1/2}} = \frac{4\pi ab^2\delta k_\prime x'}{c^3}\left(\frac{c}{a_\prime} - \frac{1}{2}\log.\frac{a_\prime + c}{a_\prime - c}\right)|$$

The value of $d\omega' \div dN_e$ being obtained in the next problem.|

ATTRACTION OF THE OBLATE ELLIPSOID OF REVOLUTION.

By equations (12.) and (19.)|

$dk_\prime \times \frac{4\pi a^2 bdc}{c(b_\prime^2+c^2)} = \frac{d\omega}{db_\prime}$ = the attraction of the oblate shell for the point at the distance $b_\prime$ from the center on| the minor axis, $\omega =$ its potential, and $a_\prime^2 y'^2 + b_\prime^2 x'^2 = a_\prime^2 b_\prime^2$, the equation of a section of its level surface through| (x',y') and the major axis $a_\prime$. Since its level surfaces have the same foci $a_\prime da_\prime = b_\prime db_\prime$. Let $dN_e =$ the infinitely| small perpendicular distance between two level surfaces at (x',y'). $dN_e = \left(1 + \frac{dy'^2}{dx'^2}\right)^{1/2} dx' = (a_\prime^4 y'^2 + b_\prime^4 x'^2)^{1/2}\frac{dx'}{dy'} = n|$ [61]

$$dy' = \frac{a^2 y'}{b^2 x'}dx' = d\left(\frac{b_\prime}{a_\prime}a'^2 - x'^2)^{1/2}\right)^{62} = d\frac{(a_\prime^2 - c^2)^{1/2}}{a_\prime}(a_\prime^2 - x'^2)^{1/2} = \frac{(a_\prime^2 - c^2)^{1/2} a_\prime da_\prime}{a_\prime(a_\prime^2 - x'^2)^{1/2}}$$

$$+\frac{(a_\prime^2 - x'^2)^{1/2} a_\prime da_\prime}{(a_\prime^2 - c^2)^{1/2} a_\prime}| - \frac{(a_\prime^2 - c^2)^{1/2}(a_\prime^2 - x'^2)^{1/2} da_\prime}{a_\prime^2} - \frac{(a_\prime^2 - c^2)^{1/2} x' d'}{a'\,(a_\prime^2 - c^2)^{1/2}},^{63} \quad \frac{a_\prime^2 y'}{b_\prime^2 x'}dx' +$$

$$\frac{b_\prime^2 x'}{a_\prime^2 y'}dx' = \frac{y' db_\prime}{b_\prime} + \frac{b_\prime^3 db_\prime}{a_\prime^2 y'} - \frac{b_\prime y' db_\prime}{a_\prime^2},|\ dx' = \frac{(c^2 y'^2 + b_\prime^4) b_\prime x' db_\prime}{a_\prime^4 y'^2 + b_\prime^4 x'^2}, dN_e = \frac{(c^2 y'^2 + b_\prime^4) db_\prime}{b_\prime(a_\prime^4 y'^2 + b_\prime^4 x'^2)^{1/2}},$$

$$\frac{d\omega}{dN_e} \times \frac{a_\prime^2 y'}{(a_\prime^4 y'^2 + b_\prime^4 x'^2)^{1/2}} = \frac{4\pi a^2 bb_\prime \delta k_\prime y' dc}{c(c^2 y'^4 + b_\prime^4)}_{64} = \text{component in| direction } y',$$

$$b_\prime = \frac{a_\prime y'}{a_\prime^2 - x'^2)^{1/2}},^{65}\ c = \frac{a_\prime(a_\prime^2 - x'^2 - y'^2)^{1/2}}{(a_\prime^2 - x'^2)^{1/2}},$$

61 RJA: for $(a_\prime^4 y'^2 + b_\prime^4 x'^2)^{1/2}\frac{dx'}{dy'}$ read $(a_\prime^4 y'^2 + b_\prime^4 x'^2)^{1/2}\frac{dx'}{b_\prime^2 x'}$.

62 RJA: after $\frac{b_\prime}{a_\prime}$, and before a'^2, insert (.

63 RJA: for d' read dx'; and in the denominator, for c^2 read x'^2.

64 RJA: for y'^4 read y'^2.

65 RJA: for $\frac{a_\prime y'}{a_\prime^2 - x'^2)^{1/2}}$ read $\frac{a_\prime y'}{(a_\prime^2 - x'^2)^{1/2}}$.

$$a_{\prime} = \frac{1}{2}(y'^2 + (x'+c)^2)^{1/2} + \frac{1}{2}(y'^2 + (x'-c)^2)^{1/2},|$$

$$da_{\prime} = \frac{(x'+c)dc}{2(y'^2 + (x'+c)^2)^{1/2}} - \frac{(x'-c)dc}{2(y'^2 + (x'-c)^2)^{1/2}} = \frac{(x'+c)dc}{2\left(a_{\prime} + \frac{cx'}{a_{\prime}}\right)} - \frac{(x'-c)dc}{2\left(a_{\prime} - \frac{cx'}{a_{\prime}}\right)}$$

$$= \frac{a_{\prime}cy'^2dc}{a_{\prime}^4 - c^2x'^2}{}^{66} = \frac{a_{\prime}cy'^2dc}{c^2y'^2 + b_{\prime}^4}.|$$

Hence, $\displaystyle\int_0^c \frac{4\pi a^2 bb_{\prime}\delta k_{\prime}y'dc}{c(c^2y'^2 + b_{\prime}^4)} = \frac{4\pi a^2 b\delta k_{\prime}}{c^3}\int_0^{(x'^2+y'^2)^{1/2}} \frac{b_{\prime}cda_{\prime}}{a_{\prime}y_{\prime}}{}^{67}$

$$= \frac{4\pi a^2 b\delta k_{\prime}}{c^3}\int_0^{(x'^2+y'^2)^{1/2}} \frac{a_{\prime}y'}{(a_{\prime}^2 - x'^2)^{1/2}} \times \frac{a_{\prime}(a_{\prime}^2 - x'^2 - y'^2)^{1/2}}{a_{\prime}^2 - x'^2)^{1/2}}|^{68} \times \frac{da_{\prime}}{a_{\prime}y'}$$

$$= \frac{4\pi a^2 b\delta k_{\prime}}{c^3}\int \frac{z^2dz}{z^2 + y'^2} = \frac{4\pi a^2 b\delta k_{\prime}}{c^3}\left(z - y'\tan.^{-1}\frac{z}{y'}\right)$$

$$= \frac{4\pi a^2 b\delta k_{\prime}}{c^3}\left((a_{\prime}^2 - x'^2 - y'^2)^{1/2} - y'\tan.^{-1}\frac{(a_{\prime}^2 - x'^2 - y'^2)^{1/2}}{y'}\right)|$$

$$= \frac{4\pi a^2 b\delta k_{\prime}y'}{c^3}\left(\frac{c}{b_{\prime}} - \tan.^{-1}\frac{c}{b_{\prime}}\right),$$

which is the component, perpendicular to the transverse axis, of the attraction of the| oblate ellipsoid of revolution.|

The component parallel to the transeverse[69] axis equals|

$$\int \frac{d\omega}{dN_e} \times \frac{b_{\prime}^2x'}{a_{\prime}^2} \times \frac{a_{\prime}^2}{(a_{\prime}^4y'^2 + b_{\prime}^4x'^2)^{1/2}} = \int_0^c \frac{4\pi a^2 b\delta k_{\prime}dcdb_{\prime}}{a_{\prime}^2c} \times \frac{b_{\prime}(a_{\prime}^4y'^2 + b_{\prime}^4x'^2)^{1/2}}{(c^2y'^2 + b'^4)} \times \frac{b_{\prime}^2x'}{a'^2}$$

$$\times \frac{a_{\prime}^2}{(a_{\prime}^4y'^2 + b_{\prime}^4x'^2)^{1/2}},|$$

$$= \int_0^c \frac{4\pi a^2 b_{\prime}^3 c\delta k_{\prime}x'dc}{a_{\prime}^2c(c^2y'^2 + b_{\prime 4})} = \frac{4\pi a^2 b\delta k_{\prime}x'}{c^3}\int_{a_{\prime}}^{(x'^2+y'^2)^{1/2}} \frac{b_{\prime}^3cda_{\prime}}{a_{\prime}^3y'^2}$$

$$= \frac{4\pi a^2 b\delta k_{\prime}x'}{c^3}\int_{a_{\prime}}^{(x'^2+y'^2)^{1/2}} \frac{a_{\prime}^3y'^3}{a_{\prime}(a_{\prime}^2 - x'^2)^3} \times \frac{a_{\prime}(a_{\prime}^2 - x'^2 - y'^2)^{1/2}}{(a_{\prime}^2 - x'^2)^{1/2}} \times \frac{da_{\prime}}{a_{\prime}^3y'^2}|^{70}$$

66 RJA: for $\frac{a_{\prime}cy'^2dc}{a_{\prime}^4 - c^2x'^2}$ read $\frac{a_{\prime}^3cy'^2dc}{b_{\prime}^2(a_{\prime}^4 - c^2x'^2)}$.

67 RJA: for y read y'.

68 RJA: for $a_{\prime}^2 - x'^2)^{1/2}$ read $(a_{\prime}^2 - x'^2)^{1/2}$.

69 'transeverse' should read 'transverse'.

70 RJA: for $b_{\prime 4}$ read $b_{\prime}^4$; and for $(a_{\prime}^2 - x'^2)^3$ read $(a_{\prime}^2 - x'^2)^{\frac{3}{2}}$.

$$= \frac{4\pi a^2 b\delta k_{\prime} x' y'}{c^3}\int_{a_{\prime}}^{(x'^2+y'^2)^{1/2}} \frac{(a_{\prime}^2 - x'^2 - y'^2)^{1/2} a_{\prime} da_{\prime}}{(a_{\prime}^2 - x'^2)^2} = \frac{4\pi a^2 b\delta k_{\prime} x' y'}{c^3}\int \frac{z^2 dz}{(z^2 + y'^2)^2}$$

$$= \frac{4\pi a^2 b\delta k_{\prime} x' y'}{c^3}\left(-\frac{z}{2(z^2 + y'^2)} + \frac{1}{2y'}\tan.^{-1}\frac{z}{y'}\right)|$$

$$= \frac{4\pi a^2 b\delta k_{\prime} x'}{c^3}\left(\frac{1}{2}\tan.^{-1}\frac{(a_{\prime}^2 - x'^2 - y'^2)^{1/2}}{y'} - \frac{(a_{\prime}^2 - x'^2 - y'^2)^{1/2} y'}{2(a_{\prime}^2 - x'^2)}\right)$$

$$= \frac{2\pi a^2 b\delta k_{\prime} x'}{c^3}\left(\tan.^{-1}\frac{c}{b_{\prime}} - \frac{b_{\prime} c}{a_{\prime}^2}\right).|$$

In which $a_{\prime}$ and $b_{\prime}$ are the semi axes of the ellipsoid, passing through the attracted point (x', y'), and having the| same foci as that of the given ellipsoid.|

A ROTATING FLUID MASS.

A fluid mass, in the form of an oblate ellipsoid of revolution, rotating about an axis with the proper angular| velocity, is in equilibrium.

First, when of homogeneous density, let R = the accelerative attraction when at rest, for any point (x', y'), on| its meridian section, the latitude of which is l. α = the angular velocity, such that R', the resultant of R and the| accelerative centrifugal force $\alpha^2 x'$, may be perpendicular to the surface.|

The ellipse gives the equations

$$y'^2 = \frac{b^2}{a^3}(a^2 - x'^2)\ (1.),\ \ \text{sine}\ l = \text{sine}\frac{R}{x}{}^{71} = \text{sine}\ \tan.^{-1}\frac{a^2 y'}{b^2 x'} = \frac{a^2 y'}{(a^4 y'^2 + b^4 x'^2)^{1/2}}\ (2.)|$$

$$x'^2 = \frac{a^4 \cos.^2 l}{(a^2\cos.^2 l + b^2 \text{sine}^2 l)}(3.),\ y'^2 = \frac{b^4 \text{sine}^2 l}{(a^2\cos.^2 l + b^2\text{sine}^2 l)}(4.),$$

$$(x'^2 + y'^2)^{1/2} = \left(\frac{\frac{a^4}{b^4}\cos.^2 l + \text{sine}^2 l}{\frac{a^2}{b^2}\cos.^2 l + \text{sine}^2 l}\right)^{\frac{1}{2}}\ (5.),|^{72}$$

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GRAVITATION| [running head]

$\tan. l = \dfrac{a^2 y'}{b^2 x'}$. The formula for the attraction are, $\dfrac{2\pi a^2 b\delta k_{\prime} x'}{c^3}\left(\tan.^{-1}\dfrac{c}{b} - \dfrac{bc}{a^2}\right) = R\cos.\dfrac{R}{x}$ (6.)| $\dfrac{2\pi a^2 b\delta k_{\prime} y'}{c^3}\left(\dfrac{c}{b} - \tan.^{-1}\dfrac{c}{b}\right) = R\text{sine}.\dfrac{R}{x}$ (7.), the components perpendicular and parallell[73] to

71 RJA: eq. (2.) for $\text{sine}\frac{R}{x}$ read $\sin\frac{R'}{x}$.

72 RJA: eq. (5.), put b as a factor in the second member.

73 'parallell' should read 'parallel'.

the axis of rotation.| Since R' is the resultant of the two forces R and $\alpha^2 x'$, there results the two equations, $R : R' :: \text{sine}{}^{R'}_{x} : \text{sine}{}^{R}_{x}$ (8.)| $R'^2 = R^2 + \alpha^4 x'^2 - 2\alpha^2 x' R\cos.{}^{R}_{x}$, (9.) Hence

$$R' = \frac{R\text{sine}\frac{R}{x}}{\text{sine }l} = \frac{4\pi a^2 b\delta k_\prime \left(\frac{c}{b} - \tan.^{-1}\frac{c}{b}\right)}{c^3(a^2\cos.^2 l + b^2\text{sine}^2 l)^{1/2}}$$

(10.).[74] By (8.), (7.), (6.),| (2.), (9.),

$$R'^2 = \frac{16\pi^2 a^4 b^2 \delta^2 k_\prime^2 y'^2 \left(\frac{c}{b} - \tan.^{-1}\frac{c}{b}\right)^2}{c^6} \times \frac{a^4 y'^2 + b^4 x'^2}{a^4 y'^2} = \frac{4\pi^2 a^4 b^2 \delta^2 k_\prime^2 x'^2}{c^6}\left(\tan.^{-1}\frac{c}{b} - \frac{bc}{a^2}\right)^2 + |^{75}$$

$$\frac{16\pi^2 a^4 b^2 \delta^2 k_\prime^2 y'^2}{c^6}\left(\frac{c}{b} - \tan.^{-1}\frac{c}{b}\right)^2 + \alpha^4 x'^2 - \frac{4\pi a^2 b\delta k_\prime x'^2\left(\tan.^{-1}\frac{c}{b} - \frac{bc}{a^2}\right)}{c^3},$$

from which there results,|[76]

$$\delta k_\prime = \frac{c^3\alpha^2}{2\pi a^2 b\left(\tan.^{-1}\frac{c}{b} - \frac{bc}{a^2}\right) - 4\pi b^3\left(\frac{c}{b} - \tan.^{-1}\frac{c}{b}\right)}, \text{ (11.)}.$$

Since $\cos.\frac{R'}{(x'^2} + y'^2)^{1/2} = \cos.\left(l - \tan.^{-1}\frac{y'}{x'}\right)$|[77]

$$= \cos.\left(l - \tan.^{-1}\frac{b^2}{a^2}\tan.l\right) = \frac{\cos.l}{\left(1 + \frac{b^4}{a^4}\tan.^2 l\right)^{\frac{1}{2}}} + \frac{\sin.l \times \frac{b^2}{a^2}\tan.^2 l}{\left(1 + \frac{b^4}{a^4}\tan.^2 l\right)^{\frac{1}{2}}} \frac{\frac{a^2}{b^2}\cos.^2 l + \text{sine}^2 l}{\left(\frac{a^4}{b^4}\cos.^2 l + \text{sine}^2 l\right)^{\frac{1}{2}}};^{78}$$

and considering $\frac{a}{b}$| constant the $\int_0^{(x'^2+y'^2)^{1/2}} R'\cos.\left(\frac{R'}{x'^2} + y'^2)^{1/2} d(x'^2 + y'^2)^{1/2} = \int_0^b \frac{4\pi a^2 b\delta k_\prime}{c^3}\right.$ $\left(\frac{c}{b} - \tan.^{-2}\frac{c}{b}\right) bdb = \frac{4\pi a^2 b\delta k_\prime}{c^3}\left(\frac{c}{b} - \tan.^{-1}\frac{c}{b}\right) \times \frac{1}{2}b^2$,| (12) which is the sum of the components in the direction of the center of R' for all points on any line from the center| to the surface. Hence the weights of all columns from the center to the surface are equal, and by hypothesis the| resultant of all forces on any part of the surface is perpendicular to it, therefore the fluid is in equilibrium at its| surface. And since the attraction of an ellipsoidal shell having its surfaces similar, concentric, and similarly placed,| is zero for an interior point, all the points of the homogeneous fluid ellipsoid are in equilibrium. And since it is|

74 RJA: eq. (10.), put b^2 as a factor in the second member.

75 RJA: and α^2 as a factor in the last term of the next equation.

76 RJA: after results, read, by making $y' = 0$.

77 RJA: for $\frac{R}{x'^2} + y'^2)^{1/2}$ read $\frac{R'}{(x'^2 + y'^2)^{1/2}}$, and in the others, $\left(\begin{matrix}\text{the same}\\ \text{the same}\end{matrix}\right)$.

78 RJA: before $\frac{\frac{a^2}{b^2}\cos.^2 l + \text{sine}^2 l}{\left(\frac{a^4}{b^4}\cos.^2 l + \text{sine}^2 l\right)^{1/2}}$ read =.

found, by tracing the curve represented by (11.), when $b \div a$ and $\delta k_{\prime}$ are its co-ordinates, that $b \div a$ has only two| values, it follows that there are only states of the equilibrium.

Second, When not of homogeneous density, let the ellipsoid of homogeneous density δ and semi axes a and b| be surrounded by a shell of density δ', having the semi axes of its outer surface $a_{\prime}$ and $b_{\prime}$ coincident with a and b| throughout their extent, and proportional to them. Then the attraction of the ellipsoid and shell for any point| will be equal to the attraction of an ellipsoid whose density is δ' plus or minus the attraction of one whose density| $= \frac{1}{2}\delta - \delta'$, according as δ' is less or greater than δ. And the formula corresponding to (10.) is|

$$R' = \frac{\frac{4\pi a_{\prime}^2 b_{\prime}\delta' k_{\prime}}{c^3}\left(\frac{c_{\prime}}{b_{\prime}} - \tan.^{-1}\frac{c_{\prime}}{b_{\prime}}\right) \pm \frac{4\pi a^2 b(\delta - \delta')k_{\prime}}{c^3}\left(\frac{c}{b_{\prime}} - \tan.^{-1}\frac{c}{b_{\prime}}\right)b_{\prime}^2}{(a^2\cos.^2 l + b^2\text{sine}^2 l)^{1/2}},$$

(13)[79]. And the equilibrium of the column| is proved as for the ellipsoid of homogeneous density. From (10.) or (13.) $\frac{R'}{R''} = \left(\frac{a^2\cos.^2 l'' + b^2\text{sine}^2 l''}{a^2\cos.^2 l' + b^2\text{sine}^2 l'}\right)^{\frac{1}{2}}$ (14)|

Hence $\frac{b^2}{a^2} = -\frac{1 - \frac{R'^2}{R''^2}}{\text{sine}^2 l'' - \frac{R'^2}{R''^2}\text{sine}^2 l'} + 1$, (15.) which is the square of the ratio semi axes[80] of a rotating fluid mass| of any law of density, when in equilibrium.|

Since $R'\cos.\left(\frac{R'}{x'^2} + y'^2\right)^{1/2} d(x'^2 + y'^2)^{1/2} = R''\cos.\left(\frac{R''}{x'^2} + y'^2\right)^{1/2} d(x''^2 + y''^2)^{1/2}$, and $d(x'^2 + y'^2)^{1/2}$: $d(x''^2 + y''^2)^{1/2}$| :: $(x'^2 + y'^2)^{1/2}$: $(x''^2 + y''^2)^{1/2}$| it follows that $R'\cos.\left(\frac{R'}{x'^2} + y'^2\right)^{1/2}(x'^2 + y'^2) = R''\cos.\left(\frac{R''}{x'^2} + y'^2\right)^{1/2}(x''^2 + y''^2)$,| (16.) showing the[81] components toward the center at any two points on any surface similar to outside surface, are as| the distances from the center.[82]

If n and n' are the lengths of arcs of the meridian, whose extremities differ in latitude by a small quantity,| say 1°, $\frac{b^2}{a^2} = 1 - e^2 = 1 - \frac{n^{2/3} - n'^{2/3}}{n^{2/3}\text{sine}^2 l'' - n'^{\frac{2}{3}}\text{sine}^2 l'}$. which compared with (15.) shows that the lengths of the seconds| pendulum are as the cube roots of the arcs of a degree, when the figure is that due to a rotating fluid mass.|

The value of $\delta k_{\prime}$, from (11.) substituted in (13.),[83] dividing both sides by $a\alpha^2$ the equatorial centrifugal force,| and making $l = 0$, and denoting by $R'e$ the equatorial force of gravity, there results| $\frac{R'e}{a\alpha^2} = \frac{\tan.^{-1}\frac{c}{b} - 3\frac{bc}{a^2} + 2\frac{b^2}{a^2}\tan.^{-1}\frac{c}{b}}{\tan.^{-1}\frac{c}{b} - \frac{bc}{a^2}}$ |[84,85]

79 RJA: for a^2 read $a_{\prime}^2$; and for b^2 read $b_{\prime}^2$.

80 RJA: for ratio semi axes read ratio of the semi axes.

81 RJA: for 'showing the' read 'showing that the'.

82 RJA: for 'center' read 'centre reciprocally'.

83 RJA: for (13.) read (10.).

84 RJA: for last equation, read $\frac{R'}{a\alpha^2} = \frac{2b^2\left(\frac{c}{b} - \tan.^{-1}\frac{c}{b}\right)}{(a^2 + 2b^2)\tan.^{-1}\frac{c}{b} - 3bc}$, as in last line [N.B. R' here should read R'_e.]

85 In the original, this equation is inserted at the bottom left of the page and the next three lines here are placed to its right.

The first member, from the equatorial radius, and the length of the| equatorial seconds pendulum as given by Prof. Airy, equals 289| nearly. Then solving the equation for $a \div b$, $1 - a \div b$ the ellip-|ticity equals about $\frac{1}{215}$[86] instead of $\frac{1}{230}$ and $\frac{1}{300}$, the results usually| given.[87]

Comments on 'Gravitation to the sphere'

Adcock's work is initially difficult to understand because it lacks an introduction, summary and final conclusions (abstracts appear to have been rarely used in scientific journal articles before 1895).[88] What is more, it contains no in-depth study nor comparison with existing work. It appears to be the culmination of several contributions by him on gravitational attraction and the flattening of the Earth, which he reported on at scientific meetings, as mentioned, for example, in his paper 'Equilibrium of a fluid mass in the form of an ellipsoid rotating about its shorter axis'.[89] He seems to have considered the pamphlet transcribed here, which he refers to as 'my work on Gravitation', to be his principal legacy to science;[90] otherwise it cannot be understood why he went to the trouble of publishing it at his own expense and announcing it wherever he might find an open ear.

In what follows we consider his findings and comment on each page of his pamphlet. In particular, we note what was known in his day and what his new contributions really were.

Pages 1 and 2

The only information as to the subject of Adcock's paper is given in its title. It concerns, first, the gravitational force between two masses of ellipsoidal shape; and, second, the ratio of the axes of a rotating fluid body which is also ellipsoidal or, more precisely, a spheroid as a result of its rotation about an axis. The latter problem concerns the 'flattening' of a celestial object, such as Jupiter or the Earth.

The historical development of how to compute the gravitational interaction between *spherical* objects, and how to predict the flattening of rotating bodies mathematically, is fully described in the monograph by Todhunter.[91] Such flattening has also been extensively analysed in a recent English translation of the classical monograph by Klein and Sommerfeld, which provides many further references.[92] However, finding explicit formulae which describe the gravitational interaction between *ellipsoidal* distributions of mass is more difficult. The PhD thesis by Schmidt provides a clue.[93] Heymann states

86 RJA: in last line but one, for about $\frac{1}{215}$ read between $\frac{1}{232.016}$ and $\frac{1}{232.698}$.; The fifth approximating fraction of $\tan.^{-1}\frac{c}{b}$ substituted in $\frac{R'_e}{a\alpha^2}$, giving $\frac{1}{232.18}$.

87 Followed in the original by an unnumbered page consisting of 39 errata, given here as footnotes.

88 Pontus Plavén-Sigray, Granville James Matheson, Björn Christian Schiffler and William Hedley Thompson, 'Research: the readability of scientific texts is decreasing over time', *eLife* (2017), https://elifesciences.org/articles/27725 (accessed 24 October 2019).

89 Robert J. Adcock, 'Equilibrium of a fluid mass in the form of an ellipsoid rotating about its shorter axis', *Am. J. Sci. Arts* ser. 3, **4**, 506 (1872).

90 *Ibid*., note.

91 Todhunter, *op. cit*. (note 30).

92 Felix Klein and Arnold Sommerfeld *The theory of the top, vol. III: perturbations. astronomical and geophysical applications* (transl. Raymond J. Nagem and Guido Sandri) (Birkhäuser, Boston; Springer, London, 2012); originally published as Felix Klein and Arnold Sommerfeld, *Über die Theorie des Kreisels, Heft III. Die störenden Einflüsse. Astronomische und geophysikalische Anwendungen* (B.G. Teubner, Leipzig, 1898).

93 Maarten Schmidt, 'A model of the distribution of mass in the galactic system', PhD thesis, Leiden University (1956); Maarten Schmidt, 'A model of the distribution of mass in the galactic system', *B. Astron. I. Neth.* **13**(468), 15–41 (1956).

that Newton had attempted to find the gravitational field of a homogeneous ellipsoidal body but did not succeed,[94] and that much later Laplace reconsidered the problem.[95] Furthermore, although Laplace's method was complex, it was not strict enough and had to be improved later by Gauss and Dirichlet as part of their development of a general potential theory.[96]

Although, in principle, Adcock could have had access to these publications, there is no evidence that he was familiar with them other than the fact that, in a footnote, he states: 'my work on Gravitation … will … contain the only correct formula ever published for determining the figure of the earth on the hypothesis of fluidity, whether that figure be an ellipsoid of three unequal axes or two'.[97] In a later comment, made in reply to a critic of his pamphlet, it appears that, when his interest in this topic began, he was only familiar with the works of Airy, Peirce, Todhunter and Pratt.[98]

We believe that Adcock's findings were a truly original discovery which have become lost in time. This is particularly so because, many years after his death, Otto Heymann, an engineer working at Siemens Apparate und Maschinen Gesellschaft in Berlin, also gave formulae for the potentials for an ellipsoid, apparently unaware of Adcock and his work.[99]

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Adcock's exposition begins with an introduction to his cumbersome nomenclature and then turns to analysing the gravitational attraction inside, and outside, a (*homogeneous*) *sphere*, as well as a *spherical shell*. His arguments are quite unwieldy and lack the elegance of modern potential theory and vector calculus.[100] Nevertheless, his final results for this case are correct: to an outside (unit) test mass, the gravitational action of the sphere or spherical shell is that of a mass of equivalent size situated in the centre, and within a spherical shell there is no attraction at all. However, these initial results are not new, as they can be found in Newton's *Principia* or in Maupertuis's (1735) work.[101]

94 Otto Heymann, 'Über das Gravitationspotential eines homogenen Ellipsoids', *Astron. Nachr.* **256**, 181–186 (1935).

95 Pierre-Simon Laplace, 'Théorie des attractions des sphéroides et de la figure des planètes', in *Histoire de l'Académie Royale des Sciences. Année 1782. Avec les mémoires de mathématique & de physique, pour la même année, tirés des registres de cette académie*, pp. 113–196 (Académie Royale des Sciences, Paris, 1785).

96 Carl Friedrich Gauss, 'Theoria attractionis corporumm sphaeroidicorum ellipticorum homogeneorum', *Methodo nova tractata. Commentationes Societatis Regiae Scientiarum Gottingensis Recentiores. II. Classis Mathematicae* [paper 3], 1–24 (1813); Peter Gustav Lejeune Dirichlet, 'Über eine neue Methode zur Bestimmung vielfacher Integrale', *Mathematische Abhandlungen der Königlichen Akademie der Wissenschaften zu Berlin. Aus dem Jahre 1839*, 61–79 (1841).

97 Adcock, *op. cit.* (note 89).

98 George Biddell Airy, 'Figure of the Earth', in *Encyclopaedia metropolitana; or universal dictionary of knowledge, on an original plan: comprising the twofold advantage of a philosophical and an alphabetical arrangement* (ed. Edward Smedley, Hugh James Rose and Henry John Rose), vol. 5, pp. 165–240 (B. Fellowes *et al.*, London, 1845); George Biddell Airy, *Mathematical tracts on the lunar and planetary theories, the figure of the Earth, precession and nutation, the calculus of variations and the undulatory theory of optics. Designed for students in the University*, 4th edn (Macmillan, Cambridge, 1858); Benjamin Peirce, *A system of analytic mechanics* (Little, Brown & Co., Boston, 1855); Isaac Todhunter, *A treatise on the integral calculus and its applications with numerous examples* (Macmillan, Cambridge, 1857); Pratt, *op. cit.* (note 33).

99 Heymann, *op. cit.* (note 94).

100 However, the latter only became common knowledge some years later, as a result of lectures given at Yale University in the 1880s by the American mathematical physicist Josiah Willard Gibbs (1839–1903); Edwin Bidwell Wilson and Josiah Willard Gibbs, *Vector analysis: a text-book for the use of students of mathematics and physics: founded upon the lectures of J.W. Gibbs* (Yale University Press, New Haven, 1901).

101 Todhunter, *op. cit.* (note 30), vol. 1, pp. 2–3, §§ 4 and 5; see also Chandrasekhar, *op. cit.* (note 9), pp. 269–302. Pierre Louis Moreau de Maupertuis, 'Sur la loix de l'attraction', in *Mémoires de l'Academie Royale des Sciences. Année 1732*, pp. 343–362 (Paris, 1735); see also Todhunter, *op. cit.* (note 30), vol. 1, p. 67, § 128.

Then follows the computation of the attraction by a *prolate ellipsoid* and an *ellipsoidal shell*, where the test mass is situated at a distance h from the centre of the major and minor axes, respectively. Adcock's results are confirmed by the formulae derived in the modern work of Heymann and Schmidt.[102] Earlier versions of the same kind of analysis were given by Maclaurin, d'Alembert, Legendre and Laplace.[103] Suffice it to say that their work also was not easy reading. As Todhunter remarked: 'Legendre's treatment of this particular case is sound but very laborious; he leaves much work to be effected by the reader, the results being given, but many of the intermediate operations being omitted. … The student will find this result is correct, but the verification will be tedious.'[104]

Unfortunately, Adcock's analysis and arguments on ellipsoidal attraction are not much easier to follow, even for those with some background in this kind of problem. Nevertheless, he seems to have been one of the very first to give an explicit answer to the problem.

Page 4

The analysis of the prolate ellipsoid and ellipsoidal shell is completed and is then followed by an analogous one for the *oblate* case. As noted above, when Adcock uses the symbol 'log.' (for example in equation 7), he actually means the natural logarithm ('ln'). It should also be noted that the integration for the prolate case leading to equation 7 can be converted into the final result for the oblate one, equation 12, by making use of the identity

$$\arctan z = \frac{1}{2i} \ln \frac{1+iz}{1-iz} \text{ with } z = \frac{\sqrt{a^2 - c^2}}{r}.$$

By using expressions such as 'sine $\genfrac{}{}{0pt}{}{R'}{x}$', Adcock seems to have adopted a nomenclature similar to that used by Peirce, in which '$\genfrac{}{}{0pt}{}{p}{x'} \genfrac{}{}{0pt}{}{p}{y'} \genfrac{}{}{0pt}{}{p}{z'}$ denote the angles which this motion makes with the three mutually perpendicular axes, called the axes of x, y, and z'.[105]

Page 5

Adcock starts this page with a passage on the attraction between shells. The reason for this is probably because only the attraction of a point-like test mass by ellipsoidal mass distribution had been considered so far. He then prepares the reader for the notion of gravitational

102 Heymann, *op. cit*. (note 94); Schmidt, PhD thesis, *op. cit*. (note 93).

103 Colin Maclaurin 'De causa physica fluxus et refluxus maris', in *Philosophiae naturalis principia mathematica. Auctore Isaaco Newtono, eq. aurato. Perpetuis commentariis illustrata, communi studio* (ed. Thomas Le Seur, & François Jacquier), vol. 3, pp. 247–282 (Barrillot & Fillii, Geneva, 1742). See also Ian Tweddle, *MacLaurin's physical dissertations* (Springer-Verlag, London, 2007), pp. 103–127; Jean le Rond d'Alembert, 'XLVII mémoire. Suite des recherches sur la figure de la Terre', in *Opuscules Mathematiques, ou Mémoires sur différens Sujects de Géométrier, de Méchanique, d'Optique, d'Astronomie, &c.*, vol. 6, pp. 161–246 (Briasson, Paris, 1773); Adrien-Marie Le Gendre (Legendre), 'Suite des recherches sur la figure des planètes', in *Histoire de l'Académie des Sciences. Année MDCCLXXXIX. Avec les mémoirés de mathématique et de physique, pour la même année. Tirés des registres de cette académie*, pp. 372–454 (Libraires de l'Académie des Sciences, Paris, 1793); Pierre-Simon de la Place (Laplace), *Théorie du mouvement et de la figure elliptique des planètes* (Ph.-D. Pierres, Paris, 1784); Pierre-Simon de la Place, *Traité de mécanique céleste. Tome premier* (Crapelet, Paris, 1798). See also Todhunter, *op. cit.* (note 30), vol. 1, pp. 139–140 (§254) and 402 (§625); vol. 2, pp. 32 (§805) and 76 (§880).

104 Todhunter, *op. cit*. (note 30), vol. 2, p. 76.

105 Peirce, *op. cit.* (note 98), p. 10.

potential (see equation 18). Because the mathematical power of vector calculus was not known to him, as Wilson and Gibbs's (1901) textbook had not yet been published,[106] this is achieved in a very crude way which can only be generalized to a general three-dimensional case (which he will need later) with great effort. Moreover, from today's didactic point of view, it is questionable why Adcock did not begin his exposition with the notion of gravitational potential for an ellipsoidal mass distribution. As a matter of fact, the potential within, and outside, an arbitrary mass distribution can be found by solving the Poisson equation, which was published more than 50 years earlier.[107] It seems that this equation, and the corresponding more abstract and fundamental meaning of gravitational potential, were unknown to Adcock, because he starts to derive an expression for it from the more intuitive integral expressions for forces, which he relates to the notion of work: 'Let the term potential $= \omega$, be the general integral of the elementary quantity of work of forces directed to and dependent on the distances to fixed centers from the attracted point.'

Page 6

This deals with the integration process involved in determining the attraction of *prolate ellipsoids* for an arbitrary external point (in contrast to the previous situation, where it was confined to a position along the plane through the equator). It ends with an explicit relation for the force component *perpendicular* to the major axis. Note Adcock's use of single and double colons in $c^2 : c^2 - x^2 :: \frac{ab^2\delta}{c} : \frac{ab^2\delta(c^2-x^2)}{c^3}$ (for example). The meaning of this notation, which was commonly in use in Europe from the mid-eighteenth century onwards, and in the United States until the beginning of the twentieth century, is as follows: 'A : B = the ratio of A to B; and A : B :: C : D = the separation of two equal ratios, *e.g.*, 3 : 12 :: 2 : 8.'[108]

Page 7

Adcock begins by concluding his analysis from page 6, and finally presents the formula for the component *parallel* to the major axis of the prolate ellipsoid. A similar analysis is then performed for the case of an *oblate ellipsoid*. In fact, both cases could have been combined by using the relationship relating arctan to ln (see the commentary on page 4, above).

He finally turns to the problem of a *rotating fluid mass* in order to determine the degree of flattening of self-gravitating bodies. He first shows that the rotation about an axis results in an oblate ellipsoid of revolution, which he calls the equilibrium shape. As an alternative, he could have said that this is the shape assumed by an incompressible self-gravitating rotating body after a stationary state has been reached.[109]

106 Wilson and Gibbs, *op. cit.* (note 100).

107 Baron Siméon Denis Poisson, 'Remarques sur une équation qui se présente dans la théorie des attractions des sphéroïdes', *Nouveau bulletin de la Société philomathique de Paris* **3**, 388–392 (1812).

108 Florian Cajori, *A history of mathematical notations. 1: notations in elementary mathematics* (The Open Court Company, London, 1928), p. 294.

109 The choice of the word 'equilibrium' might require some further explanation, since it could be confused with 'stability' of the spheroidal configuration. Following and (in part) based on the work of G. C. Jacobi ('Über die Figur des Gleichgewichts', *Annalen der Physik* **109**(8–16), 229–233 (1834)) and J. Liouville ('Formules générales relatives à la question de la stabilité de l'équilibre d'une masse liquide homogène douée d'un mouvement de rotation autour d'un axe', *Journal de Mathématiques Pures et Appliquées*, ser. 1, **20**, 164–184 (1855)), the issue of instabilities during revolution of fluid, self-gravitating spheres was addressed in a seminal paper by Poincaré ('Sur l'équilibre d'une masse fluide animée d'un mouvement de rotation', *Acta Mathematica* **7**(1), 259–380 (1885)). There it

His analysis seems clumsy in comparison with today's standards, based on stringent continuum mechanics.[110]

Page 8

Finally, Adcock gives an expression which allows him to calculate a value for the flattening: $f = 1 - \frac{1}{\lambda}, \lambda = \frac{a}{b} > 1$, where a is the equatorial axis and b the polar axis of the rotating spheroid. Of paramount importance in the calculation is the value 289 taken for the ratio between the gravitational and the centrifugal acceleration at the equator:

$$\frac{m/(Ga^2)}{a\omega^2} = 289,$$

where m denotes the mass of the fluid mass subjected to self-gravitation, rotating at an angular speed ω, and G is the universal constant of gravitation. This was given in a paper by Airy and it serves as input on the left-hand side of Adcock's last formula on this page.[111] In fact, writing Adcock's formulation in modern terminology gives:

$$\frac{R_e'}{a\alpha^2} \equiv \frac{m/(Ga^2)}{a\omega^2}.$$

Note that his original equation is faulty and should be replaced by the final equation in his errata list. Its subsequent evaluation can only be performed numerically. If evaluated using a modern software tool such as *Mathematica*,[112] the flattening comes out to be 1/232.186. This agrees exactly with the value given in the last item in Adcock's list of errata. Comparison with the other two estimates given in the penultimate line of his corrections suggests that he must have obtained it by means of an iterative calculation.

Is his result for the flattening really physically significant? Consider the simple formula originally obtained by Newton, who found the ratio between the gravitational and the centrifugal acceleration to be:

$$\frac{m/(Ga^2)}{a\omega^2} = \frac{5}{4f}.$$ [113]

By replacing the left-hand side of this expression (the ratio between the gravitational and the centrifugal acceleration at the equator) with Airy's experimentally determined value 289, one obtains $f = \frac{5}{4\times 289} = \frac{1}{231.2}$, which is not very different from Adcock's result. Moreover, if the theoretical result derived by Müller:

$$\frac{m/(Ga^2)}{a\omega^2} = \frac{3}{2}\frac{(1+2\lambda^2)\arccos(\lambda) - 3\lambda\sqrt{1-\lambda^2}}{(1-\lambda^2)^{\frac{3}{2}}},$$

is shown that shapes different from spheroids may result depending on angular velocity. However, in the context of planets this has never been observed, so that all of this remains more of a mathematical curiosity, at least at small angular speeds.

110 See, for example, the arguments in the lecture notes by Richard Fitzpatrick ('Fluid mechanics, lecture notes', University of Texas at Austin, 2016, p. 45, http://farside.ph.utexas.edu/teaching/336L/Fluidhtml/node35.html (accessed 24 October 2019)), which make it immediately obvious that the symmetric ellipsoid, namely a spheroid, is the stationary equilibrium shape for an incompressible fluid body subjected to rotation about an axis.

111 Airy, *op. cit.* (note 33), p. 355.

112 Wolfram Research, Champaign, Illinois; first released in 1988.

113 Chandrasekhar, *op. cit.* (note 9), p. 384.

is evaluated numerically, one finds $f = \frac{1}{231.99}$ and comes somewhat closer to Adcock's value.[114] However, experimental findings indicate that the true flattening value for planet Earth is $f = \frac{1}{298.24}$ (see figure 1).[115] In other words, the Earth looks much less like a 'jelly donut' than is predicted from the fluid model. It is stiffer because it also contains solid layers and, in fact, this makes the whole controversy as to which value is correct a purely academic argument.

A CONTEMPORARY CRITIC

Anonymous criticism of Adcock's work, presumably by an early purchaser of his pamphlet, appeared in the *Chicago Tribune* on 30 March 1872:

> It has long been believed and taught that the flattening of the earth in the polar regions is due to the fact that the force of gravity at the equator is partially counteracted by the centrifugal force generated by the daily rotation of the earth on her axis. It is also held that this flattening, which amounts to about one part in 300, shows that the centrifugal force of the equator is equal to about one part in 289 of the force of attraction there.
>
> Mr. R.J. Adcock, of Monmouth, Ill. has written a pamphlet, the object of which is to prove that these conclusions are erroneous. He shows, by a formidable array of quantities, stated in the language of the Calculus, that the ratio of these two forces is between 373.12 and 373.76 instead of 289,[116] and that, therefore, either the earth has not the form due to a rotating fluid mass, or that the lengths of its equatorial and polar diameters have not been ascertained.[117]

The comment in the first paragraph above regarding centrifugal force refers to Newton's deduction (see above) that $\lambda = \frac{1}{289}$. Adcock replied in a letter to the same newspaper on 15 April:

> I have to say it is definitely known … that, the form of the surface of the earth not being that of an exact water-level of equilibrium, the tendency of gravity, including centrifugal force, together with the dissolving effects of the fluids on the surface, is to produce that exact form of equilibrium of the surface; and, that the equatorial diameter of this surface of equilibrium will be greater, other things being the same, with centrifugal force than it would be without rotation. Then the flattening, or a part of it, is either due to, or maintained or increased by, rotation, according as the surface differs from one of equilibrium.
>
> The one part in 300, and the one part in 289, are both deduced by correct methods from actual observation and measurement, and therefore I do not undertake to dispute these results. What I have proved is, that the so universally affirmed proposition, that the agreement of these two results with the hypothesis that the earth has the approximate form due to fluidity, is not the case.
>
> Your conclusion that the flattening of about one part in 300 shows a ratio of one part in 289 of the centrifugal force to gravity, is certainly very wild: since, except in the case of fluidity, the rotation between the flattening and this ratio could not be determined

114 Müller, *op. cit.* (note 26).

115 Williams, *op. cit.* (note 32).

116 It is not clear where the figures of ~373 in this correspondence come from. They are not in the printed copy of Adcock's pamphlet nor in any newspaper article or correspondence which it has been possible to trace, but the present-day collections of scanned newspapers do not, unfortunately, provide a complete record.

117 *Chicago Tribune*, 30 March 1872, p. 4.

theoretically, even in the case of the perfect equilibrium of the surface, without knowing the internal constitution of the earth. And, in the case of perfect fluidity, I have shown—from formulae … that this flattening of one part in 300, and the ratio of 289, are widely inconsistent with the condition of fluidity.

Now, as to my object in writing the treatise: It is to show for the benefit of science and myself, the accomplishment of what is mentioned in the title-page, to find the 'Attraction to the sphere and the two ellipsoids of revolution,' and to find the 'ratio of the axes of a rotating fluid mass' under the action of the gravity of its own particles, according to the universal law of Newton.

When I adopted my method of treating attraction, I was acquainted only with that given in Professor Pierce's [*sic*] *Analytical* [*sic*] *Mechanics*,[118] which I thought might be rendered more elementary; and, after examination of Professor [I.] Todhunter's, J.H. Pratt's and Professor Airy's methods, I considered mine, if not shorter, at least more easily understood, original to some extent, and American, accomplishing by direct integration what J.H. Pratt said was 'impossible to integrate by any known method of direct integration'.[119]

And, after correctly solving the problem of the form and ratio of the axes of a rotating fluid mass under the action of the gravity of its own particles, I was convinced that this problem, though attempted by Newton, and held since 1743[120] to have been solved by Clairaut, Jacobi, Laplace, Bowditch, Airy, *etc*., *etc*., was correctly solved for the first time, though I was not able then, and not until quite recently, to show what was the error in the reasoning of others on this subject, which is simply in violating the axiom that, in the same calculation, it will not do to use as equal two different units of measure of the same or similar quantities … [these] are two differing measuring units of force,—one the attraction of a unit of mass at the unit of distance, the other the centrifugal force of a unit of mass revolving with a unit of velocity in a circle of radius unity.[121]

Adcock completed the arguments for his proof that 'a fluid mass, in the form of an ellipsoid, rotating about its shortest axis, under the action of the attraction of its own particles and their centrifugal forces, is in equilibrium; and this is the only form of equilibrium' in a short note (which contains no numerical results) published in December 1872.[122] However, his publications on the topic appear not to have attracted attention subsequently. It is probable that his pamphlet appeared too late to be included in Todhunter's definitive survey of publications on 'the figure of the earth' (since its preface is dated July 1873).[123] In any case, by that time interest in the problem had essentially lapsed. Adcock's pamphlet became effectively lost thereafter and is transcribed here for the first time.[124]

Conclusions

Adcock's 1872 paper on the figure of the Earth gives a complete, if somewhat crude, derivation of the solution of a problem first considered, but not solved, by Newton,

118 Peirce, *op. cit.* (note 98).
119 This is an inexact quotation from Pratt, *op. cit.* (note 33), p. 10.
120 Alexis Claude de Clairaut, *Theorie de la figure de la Terre, tirée des principes de l'hydrostatique* (David Fils, Paris, 1743).
121 *Chicago Tribune*, 15 April 1872, p. 5.
122 Adcock, *op. cit.* (note 89).
123 Todhunter, *op. cit.* (note 30).
124 See Müller, *op. cit.* (note 26) for a modern view of the subject.

Laplace, Gauss and Dirichlet: namely, finding the gravitational field of a rotating self-gravitating homogeneous ellipsoidal fluid mass. After lengthy calculations, all the more remarkable as they were undertaken without the benefits of potential theory and vector algebra, he obtained a formula which, until now, was believed to have first been derived by Heymann in 1935. Using Airy's 1856 value for the ratio between the gravitational and centrifugal acceleration at the equator, Adcock obtained a value for the inverse flattening of 232.19, which is very close to the 'exact' value of 231.99, obtained using modern analytic techniques applied to a model of a self-gravitating sphere made of an incompressible fluid which starts spinning about a fixed axis at a constant angular velocity and which, a result of centrifugal accelerations and internal friction, finally reaches a state of axisymmetric deformation in which it assumes the shape of a spheroid.[125] The numerical value that Adcock hand-calculated for the flattening achieved under the model is extremely close to that obtained using Wolfram's *Mathematica* software today. Curiously, he concluded his paper with the comment 'the ellipticity equals between $\frac{1}{232.016}$ and $\frac{1}{232.698}$ instead of $\frac{1}{230}$ and $\frac{1}{300}$ the results usually given', without pointing out that his work applied to an entirely theoretical model, and that the difference between his result and the real-world observed value of 298.24 for the Earth is accounted for by the fact that the latter has a more complex interior structure.

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125 *Ibid.*