On the simple group of order 25920

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Explicit expressions for the operations of the simple group of order 25920 as products of five or fewer operations of period two are obtained, leading to a determination of the conjugate sets of operations contained in the group. The results are applied to the determination of the nature of certain important subgroups, and to some questions relating to the generation of the group by two operations.

INTRODUCTION

The simple group $G$ of order 25920 has formed the subject of a large number of investigations. It was first considered by Jordan (1870) as the group of even substitutions of the 45 tritangent planes of a general cubic surface. Jordan proved that the group was simple, and identified it with the group arising in the problem of trisection of the periods of quadruply periodic hyperelliptic functions. From this latter point of view the group was studied at length by Burkhardt, who obtained various representations of the group by linear transformations. In particular, he showed (Burkhardt 1890) that the group is simply isomorphic with a collineation group in space of four dimensions leaving a certain quartic primal invariant, and obtained a set of four particular operations which generate the group. The primal in question possesses 45 nodes, which are in (1,1) correspondence with the 45 tritangent planes of a general cubic surface, and form a remarkable configuration. The geometry of this configuration has been developed by Baker in his recent tract (Baker 1946). Baker has drawn attention to the importance of 45 particular operations of the group, which he has termed projections, each of which is a harmonic inversion having as centre one of the 45 nodes. He proves, among many other interesting results, that Burkhardt’s four generators can all be expressed in terms of these projections, which accordingly generate the whole group.

The importance of these projections rests on the fact, proved below, that they form a complete set of conjugate operations in $G$. It will be proved in the present paper that every operation of the group can be expressed as a product of five or fewer of these projections, and shall express every operation of the group in such a form. It will be seen how this method of expressing the operations enables one to enumerate the various conjugate sets belonging to the group, confirming the results obtained (from a very different point of view) by Frame (1936); to determine readily the periods of the different types of operation; to enumerate the operations of certain important subgroups of $G$; and to verify results obtained by Coxeter, Brahma and Frame (referred to in §8 below) concerning various simple ways in which $G$ can be generated by a small number of operations, and to obtain other results of a similar nature.
1. Description of a fundamental configuration in four dimensions

This paper is concerned with the group $G$ of collineations in four dimensions which leaves invariant a certain configuration of 45 points, and it is the purpose of the present section to describe this figure, and certain subsidiary figures contained in it, in sufficient detail to make the rest of the paper intelligible. Proofs of all the statements made here will be found in Baker's tract (Baker 1946), or may easily be deduced from Baker's results.

Take a coordinate system in $S_4$ in which a point is specified by the ratios of six numbers $x_1, x_2, ..., x_6$ whose sum is zero, and consider the $30 + 15$ points $(ij, kl, mn), (ij)$ defined, respectively, by

\[ x_i = x_j = 1, \quad x_k = x_l = e, \quad x_m = x_n = e^2; \]

and

\[ x_i = 1, \quad x_j = -1, \quad x_k = x_l = x_m = x_n = 0, \]

where $(ijklmn)$ is a permutation of $(123456)$ and $e$ is a primitive cube root of unity. It is easily seen that

\[(ij, kl, mn) = (ji, kl, mn) = (kl, mn, ij) = etc., \quad (ij) = (ji).\]

These 45 points are the nodes of the quartic primal whose equation is

\[ \sum x_i x_j x_k x_l = 0, \]

the summation extending over the 15 sets of four taken from among the suffixes $1, 2, ..., 6$; and it will be convenient to refer to them throughout the paper as nodes, although explicit reference to the primal will not be made in the sequel.

The group $G$ of collineations, which keeps the set of nodes as a whole fixed, permutes the nodes transitively (that is, it contains operations which change any given node into any other), and it also permutes transitively the different configurations derived from these nodes which will now be described.

With each node is associated a definite prime, which will be called the polar prime of the node. The polar primes of the nodes $(ij, kl, mn), (ij)$ have respective equations

\[ (x_i + x_j) + e^2(x_k + x_l) + e(x_m + x_n) = 0, \quad x_i - x_j = 0. \]

The 45 primes so arising will be called, following Baker, Jordan primes.

With respect to any given one of the 45 nodes, $A$ say, the remaining nodes fall into two sets, comprising 12 and 32 nodes respectively. The 12 nodes in the former of these two sets lie in the polar prime of $A$, and the line joining any one of these to $A$ contains no further node. Such lines will be called $e$-lines. There are $\frac{1}{2} \cdot 45 \cdot 12 = 270$ $e$-lines, each containing two of the nodes, 12 of these lines passing through each node. The 32 nodes which do not lie on $e$-lines through $A$ lie in pairs on 16 lines through $A$. Such lines, which are called $k$-lines, each contain three nodes, and there are $\frac{1}{2} \cdot 45 \cdot 16 = 240$ of them, sixteen passing through each node. The polar primes of
three nodes lying on a $\kappa$-line meet in a plane containing a second $\kappa$-line; two such $\kappa$-lines are symmetrically related and either is called the polar $\kappa$-line of the other. A pair of polar $\kappa$-lines consists, necessarily, of two skew lines, and the nine lines joining a node on one $\kappa$-line to a node on its polar are all $e$-lines.

It will be important in what follows to be able to recognize whether two nodes whose symbols are given are joined by an $e$-line or a $\kappa$-line, and in the latter case to determine the third node which lies on the $\kappa$-line. To this end the sets of three nodes which lie on $\kappa$-lines are listed; all such sets have one of the three forms

\[(ij), (jk), (ki);\]
\[(ij), (ik, jl, mn), (jk, il, mn);\]
\[(ij, kl, mn), (il, kn, mj), (in, kj, ml).\]

It will be noticed, in particular, that the 12 nodes lying in the polar prime of $(ij)$ are these—other than $(ij)$—whose symbols are unaltered by the interchange of $i$ and $j$, while pairs of nodes collinear with $ij$ are such that their symbols are interchanged when $i$ and $j$ are interchanged.

There are planes, of four distinct types, each containing three or more nodes which are not collinear, as follows:

(i) There are 270 planes, which will be called $f$-planes, each containing three of the nodes and the three lines, which are $e$-lines, joining them in pairs. Each of these planes is the intersection of the polar primes of two other nodes, uniquely determined by the plane, whose join is an $e$-line. These nodes, together with the three that lie in the plane, form a symmetrical set of five nodes such that the polar prime of any one of them contains the other four; the join of any two of these nodes is an $e$-line and the plane containing any three is a $c$-plane. Such a set of five nodes is called a Jordan pentahedron, and there are 27 such, three of which contain any given node. In particular, the five nodes

\[(ij), (kl), (mn), (ij, kl, mn), (ij, mn, kl)\]

form a Jordan pentahedron, a fact which will be used in the sequel.

(ii) There are 720 planes, called $d$-planes, each of which joins a $\kappa$-line to a node lying on its polar $\kappa$-line. Each $d$-plane contains four nodes, three $e$-lines and one $\kappa$-line.

(iii) There are 540 planes, called cross-planes or $c$-planes, each containing six nodes, which are the vertices of a quadrilateral formed by $\kappa$-lines. Each $c$-plane contains six nodes, three $e$-lines (the diagonals of the quadrilateral) and four $\kappa$-lines.

(iv) There are 40 planes, called Jacobian planes or $j$-planes, each of which contains nine of the nodes, and twelve $\kappa$-lines. The nine nodes form the well-known configuration formed by the inflexions of a pencil of plane cubic curves; the join of any two of them is a $\kappa$-line and contains a third node, and the four $\kappa$-lines which pass through any one of the nodes form an equianharmonic pencil. The 12 $\kappa$-lines can be divided into four sets of three, each set forming a triangle whose sides contain all the nine nodes.
There are likewise primes, of four types, each containing four or more nodes which are not coplanar, as follows:

(i) It can easily be shown that each cross-plane lies in a definite Jordan prime, and that the polar $\kappa$-lines of the four $\kappa$-lines which lie in the plane pass through the pole of the prime (i.e. the node whose polar prime is the Jordan prime in question). The 540 primes, called $x$-primes, which join the cross-planes to the poles of the Jordan primes in which they lie, each contain seven nodes, nine $\varepsilon$-lines, four $\kappa$-lines, three $f$-planes, four $d$-planes and one $c$-plane.

(ii) There are 216 primes, called $n$-primes, each containing ten nodes and ten $\kappa$-lines which are the vertices and edges of a pentahedron of cross-planes. Each $n$-prime contains 10 nodes, 15 $\varepsilon$-lines, 10 $\kappa$-lines, 10 $d$-planes and five $c$-planes. The prime containing the ten nodes

\[(12), (13), (14), (15), (23), (24), (25), (34), (35), (45)\]

is an example.

(iii) The 45 Jordan primes already mentioned each contain 12 nodes, which fall into three sets of four, forming a triad of desmic tetrahedra, any two of which are in perspective from each vertex of the third. Each of these tetrahedra, together with the pole of the prime, forms a Jordan pentahedron. Each Jordan prime contains 10 nodes, 18 $\varepsilon$-lines, 16 $\kappa$-lines, 12 $f$-planes and 12 $c$-planes.

(iv) There are 40 Steiner primes, each of which contains four Jacobian planes which form a tetrahedron whose pairs of opposite edges are polar $\kappa$-lines; each edge of the tetrahedron thus contains three nodes. A Steiner prime contains 18 nodes, 27 $\varepsilon$-lines, 42 $\kappa$-lines, 18 $d$-planes, 27 $c$-planes and four $j$-planes.

Elementary combinatorial considerations show that the lines, planes and primes just enumerated comprise all the linear spaces defined by subsets of the 45 nodes. Some further properties of the configuration are added which will be of use later.

There are four Steiner primes containing any given Jacobian plane. Each of the four triangles formed from three $\kappa$-lines in the plane which contain between them the nine nodes is a face of the tetrahedron of Jacobian planes in one of these primes. And any prime which contains a Jacobian plane and a node not lying in this is a Steiner prime. If $\Sigma$ is a given Steiner prime, there are 12 other Steiner primes whose intersection with $\Sigma$ is a Jacobian plane. The remaining 27 Steiner primes meet $\Sigma$ in cross-planes, one such prime passing through each of the cross-planes in $\Sigma$. And the 27 Jordan primes which contain the cross-planes of $\Sigma$ are all distinct, and are the polar primes of the 27 nodes which do not lie in $\Sigma$.

From the 27 Jordan pentahedra it is possible, in 72 ways, to select a set of six pentahedra whose vertices are all distinct. Each such set determines uniquely a second set whose pentahedra contain, collectively, the same 30 vertices. Thus there are 36 double-sizes of pentahedra. The 15 nodes which are not included among the vertices of a double-six are the vertices of a hexahedron of $n$-primes, and the (ten) pairs of opposite edges of the hexahedron are polar $\kappa$-lines. Such a set of nodes, of which there are 36, may be called a 15-set, a simple example being provided by the 15 nodes (ij).
2. The Simple Group $G$ and the Projections Which Generate It

It is known (Coble 1906, p. 365) that it is possible to set up a $(1, 1)$ correspondence between the 45 nodes of the figure described in §1 and the 45 tritangent planes of a general cubic surface in such a way that the tritangent planes corresponding to two nodes $A, B$ meet in a line of the cubic surface if $AB$ is an $e$-line and have no line of the surface in common if $AB$ is a $k$-line. It follows that the group $G^*$ of automorphisms of the 45 nodes (that is, the group of permutations of the nodes among themselves which preserve incidence relations) is simply isomorphic with the group of automorphisms of the tritangent planes. This group was studied by Jordan, who showed (Jordan 1870) that it contains a self-conjugate subgroup $G$ which is simple and of order 25920; this group consists, in fact, of the group of automorphisms which effect an even permutation of the 45 tritangent planes (Dickson 1901, p. 306). I have shown elsewhere (Todd 1936, p. 173) that every operation of $G^*$ corresponds to a collineation or anticollineation in $S_4$ which transforms the set of nodes into itself. The subgroup formed by the collineations belonging to $G^*$ is of index 2, and it is clearly self-conjugate. Since $G^*$ is known to contain only one such subgroup it follows that the group of collineations, of order 25920, is simply isomorphic with $G$ (and will in future be denoted by $G$).

Baker has shown in his tract (Baker 1946, §7) that the configuration is left unaltered by the harmonic inversion with respect to any node $A$ and its polar prime. This harmonic inversion, which will be denoted by $p(A)$ and called the projection from the node $A$, is an operation of $G$ which leaves unaltered the node $A$ and the 12 nodes in the polar prime of $A$, and which interchanges in pairs the two nodes other than $A$ on each of the 16 $k$-lines which pass through $A$. Thus $p(A)$ effects an even permutation of the 45 nodes.

Suppose, now, that $B, C, D$ are further nodes such that $B$ and $C$ lie on a $k$-line through $A$, and $D$ lies in the polar prime of $A$. Let $T$ be any operation of $G$ and let $A', B', C', D'$ be the nodes into which $A, B, C, D$ are respectively transformed by $T$. Then, since $T$ is an automorphism of the configuration, $A'B'C'$ is a $k$-line and $D'$ lies in the polar prime of $A'$. Let $T' = Tp(A)T^{-1}$ be the transform of $p(A)$ by $T$. Then, since $p(A)$ leaves $A$ and $D$ unaltered and interchanges $B$ and $C$,

$$T'[B'] = Tp(A)T^{-1}[B'] = Tp(A)[B] = T[C] = C',$$
$$T'[C'] = Tp(A)T^{-1}[C'] = Tp(A)[C] = T[B] = B',$$
$$T'[D'] = Tp(A)T^{-1}[D'] = Tp(A)[D] = T[D] = D',$$

so that $T'$ keeps fixed $A'$ and every node in its polar prime, and interchanges the two nodes, other than $A'$, on each $k$-line through $A'$. Thus $T'$ is the projection $p(A')$, for distinct operations of $G$ effect distinct permutations of the nodes. It has thus been proved that

$$Tp(A)T^{-1} = p(A'), \quad \text{where} \quad A' = T[A].$$
Thus any operation of $G$ which is conjugate to a projection is itself a projection. And, since $G$ is transitive, there exist operations of $G$ which transform $A$ into any one of the other nodes. Thus all the 45 projections are conjugate. Hence the 45 projections form a complete conjugate set in $G$. The subgroup generated by them is thus self-conjugate in $G$, and hence, as $G$ is simple, coincides with $G$ itself. Thus Baker's statement that $G$ is generated by projections is verified; and since each projection effects an even permutation of the nodes the same will be true for each operation of $G$. Observing, further, that the anticolon-lineation $x' = x$ keeps fixed the 15 nodes $(ij)$ and interchanges the 15 pairs of nodes such as $(ij, kl, mn)$, $(ij, mn, kl)$, thus effecting an odd permutation of the nodes, it is seen at once that every anticolon-lineation in $G^*$ (which is the product of the one considered with a colon-lineation of $G$) effects an odd permutation of the nodes, so that every even automorphism is determined by a colon-lineation.

The fundamental relations connecting the various projections are given by Baker (1946, §7) and may be expressed in the following form (in which the identical operation of $G$ is denoted by 1):

\[
\begin{align*}
[p(A)]^2 & = 1 \text{ for each node } A, \\
p(A) p(B) p(A) & = p(B) & \text{if } AB \text{ is an e-line,} \\
p(A) p(B) p(C) & = p(C) p(A) & \text{if } ABC \text{ is a } \kappa\text{-line}, \\
p(A) p(B) p(C) p(D) p(E) & = 1 & \text{if } ABCDE \text{ is a Jordan pentahedron.}
\end{align*}
\]

I propose to show how, by means of these relations, every operation of $G$ can be expressed as a product of five or fewer projections, to classify these operations and obtain their periods, and to determine the various conjugate sets contained in $G$. It will be seen that there are fifteen types of operation, the operations of any one type all being conjugate in the group of automorphisms $G^*$, and that in the case of five of these types the operations form two conjugate sets in $G$. For purposes of reference, these types will be denoted by Roman numerals, and I shall commence with two types whose existence is already clear, namely,

I. The identical operation.

II. The 45 projections, each of period 2, which form a conjugate set.

In determining the arrangement of the operations of $G$ in conjugate sets frequent use will be made of the principle that if a set of nodes $A_1, \ldots, A_n$ is transformed by an operation $T$ of $G$ into a set $A'_1, \ldots, A'_n$, then the operations $p(A_1) \ldots p(A_n)$ and $p(A'_1) \ldots p(A'_n)$, products of the projections from these sets of nodes, are conjugate, a result which is immediately evident if one observes that

\[
Tp(A_1) \ldots p(A_n) T^{-1} = Tp(A_1) T^{-1} \ldots Tp(A_n) T^{-1} = p(A'_1) \ldots p(A'_n),
\]

by the result proved above. Two sets of nodes $A_1, \ldots, A_n$ and $A'_1, \ldots, A'_n$ related in this way will be called equivalent with respect to $G$, and the aggregate of operations conjugate to a product of projections $p(A_1) \ldots p(A_n)$ consists of the products $p(A'_1) \ldots p(A'_n)$ of projections from all sets of nodes equivalent to $A_1, \ldots, A_n$. 

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In the course of subsequent work it will frequently be necessary to transform products of projections, usually with the aim of simplifying the product under discussion or of reducing it to some standard type. The transformations are based entirely on equations (1). But one particular type of transformation is of such frequent occurrence that it is described explicitly here. From (1) it follows that any product \( p(A)p(B) \) of two projections can be expressed in each of the forms \( p(B)p(A') \) or \( p(B')p(A) \), where, if \( AB \) is an \( e \)-line, \( A' = A \) and \( B' = B \), while if \( AB \) is a \( \kappa \)-line, \( A' = B' = C \), where \( C \) is the remaining node of the \( \kappa \)-line. It follows that in any product of projections any given factor \( p(A) \) may be moved one or more places to the right (or left) without altering the number of projections in the product and without altering any factor which is not passed over by \( p(A) \). More generally, two or more factors may be moved about in the same way (with consequent changes only in the factors passed over) so long as they retain their original order. Any transformation of this type will be referred to as an *elementary transformation* of the product, and it will be assumed that such transformations will be recognized as such by the reader after a few examples have been given.† The other transformations involve either the cancelling of two consecutive factors \( p(A), p(A) \), which calls for no comment, or the replacement of the product of projections from a set of nodes belonging to a Jordan pentahedron by the product of projections from the remaining nodes of the pentahedron. When this is likely to cause any difficulty to the reader a footnote has been added by way of explanation.

By way of example two elementary results are proved. Calling a product of projections *reducible* if it can be expressed as a product involving fewer factors the following theorems are proved:

(i) *Any product which contains a repeated factor is reducible.*

(ii) *Any product which contains three distinct factors \( p(A), p(B), p(C) \)—not necessarily consecutive—is reducible if either \( ABC \) is a \( \kappa \)-line or \( ABC \) is an \( f \)-plane.*

To prove (i) it is sufficient to observe that by elementary transformations the product can be transformed into one in which the repeated factor \( p(A) \) occurs as two consecutive factors; and then, since \( p(A) \) has period 2, these two factors can be removed altogether. Similarly, in case (ii), elementary transformations will reduce the product to one in which \( p(A), p(B), p(C) \) are consecutive factors, suppose in this order. Then, if \( ABC \) is a \( \kappa \)-line, \( p(A)p(B)p(C) = p(B)p(C)p(C) = p(B) \), and if \( ABC \) belong to a Jordan pentahedron \( ABCDE \), then, by the last equation of (1),

\[
p(A)p(B)p(C) = [p(D)p(E)]^{-1} = p(E)p(D),
\]

and in each case the number of factors has been reduced.

### 3. Products of two projections

In considering the products of projections from two distinct nodes \( A, B \) there are two cases to be distinguished, according as \( AB \) is an \( e \)-line or a \( \kappa \)-line.

If \( AB \) is an \( e \)-line, equations (1) show that \( p(A) \) and \( p(B) \) commute, so that from each \( e \)-line a single product \( p(A)p(B) \) of this type arises, and this operation is of

† The list of types of \( \kappa \)-line in § 1 will be found useful here.
period 2. Thus \( p(A) \) and \( p(B) \) generate an Abelian group of order 4, direct product of the cyclic groups generated by \( p(A), p(B) \). And it is easy to see that \( p(A) p(B) \) is an axial homology having for invariant spaces the line \( AB \) and the opposite plane \( CDE \) of the Jordan pentahedron of which \( AB \) is an edge. Thus \( p(A) p(B) \) leaves the nodes \( A, B, C, D, E \) unchanged, and only these nodes. Whence it is easily seen that the operations of this type arising from different \( e \)-lines are distinct. And since \( G \) permutes the \( e \)-lines transitively, every such operation is conjugate to \( p(A) p(B) \). Hence, since there are 270 \( e \)-lines, \( G \) contains

III. A conjugate set of 270 operations of period 2, products of pairs of projections from nodes lying on an \( e \)-line.

If \( AB \) is a \( \kappa \)-line, and if \( C \) is the third node lying on \( AB \), then, from (1),

\[
p(A) p(B) = p(B) p(C) = p(C) p(A),
\]

and so

\[
[p(A) p(B)]^3 = p(A) p(B) . p(B) p(C) . p(C) p(A) = 1,
\]

so that \( p(A) p(B) \) is of period 3. Its square (or inverse) is \( p(B) p(A) \), and there are just two products of pairs of projections from nodes on \( ABC \) which are distinct; the projections \( p(A), p(B), p(C) \) generate a group of order 6, which, as the reader will easily verify, is simply isomorphic with the symmetric group of degree 3 (and in fact permutes the nodes \( A, B, C \) according to the operations of this group). The operation \( p(A) p(B) \) keeps fixed the three nodes of the polar \( \kappa \)-line of \( ABC \), and no other node. Hence the operations arising from distinct \( \kappa \)-lines are distinct. Since \( G \) is transitive on the \( \kappa \)-lines, any operation of this type is conjugate to one of the two operations \( p(A) p(B), p(B) p(A) \). And these are clearly conjugate, since each is transformed into the other by \( p(A), p(B) \) or \( p(C) \). Thus, since there are 240 \( \kappa \)-lines, \( G \) contains

IV. A conjugate set of 480 operations of period 3, each the product of projections from two nodes of a \( \kappa \)-line.

4. Products of projections from coplanar sets of nodes

4.1. Next consider the products of projections from sets of coplanar nodes. In examining such products one may, by the theorems proved at the end of §2, ignore products containing repeated factors or products involving projections from three collinear nodes. There are four cases to consider, since there are four types of plane containing three or more non-collinear nodes. In the case of the \( f \)-plane or the \( d \)-plane the results are so easily seen that they are stated without proof, their verification being left to the reader.

No new type of operation arises from an \( f \)-plane, since the product of the three projections from the nodes lying in it is reducible (§2). The group generated by the projections is an Abelian group of order 8, direct product of three cyclic subgroups of order 2.
In the case of a $d$-plane, joining a $\kappa$-line $ABC$ to a node $P$ on its polar $\kappa$-line, it is easily seen that the projections from $A, B, C, P$ generate a group of order 12, the direct product of the group of order 6 generated by $p(A), p(B), p(C)$ and the cyclic group generated by $p(P)$, the projection $p(P)$ commuting with every operation of this group. There are just two operations of a new type, namely, $p(A)p(B)p(P)$ and its inverse $p(B)p(A)p(P)$, and these operations are conjugate, being transformed into each other by $p(A), p(B)$ or $p(C)$. The operation $p(A)p(B)p(P)$ is of period 6, its square being $p(B)p(A)$ and its cube $p(P)$, so that different $d$-planes give different operations. Thus, since there are 720 $d$-planes, $G$ contains

V. A conjugate set of 1440 operations of period 6, products of three projections, whose squares are of type IV and whose cubes are of type II.

4.2. Next consider the operations derived from sets of nodes in a $c$-plane. Such a plane, as was seen in §1, contains six nodes forming the vertices of a complete quadrilateral, whose sides are $\kappa$-lines and whose diagonals are $e$-lines. Denote these nodes by $A, B, C, A', B', C'$ in such a way that the sides of the quadrilateral are $A'B'C, AB'C, ABC', A'B'C'$ and the diagonals are $AA', BB', CC'$.

It can be shown, in a variety of ways, that the projections from the six nodes in a $c$-plane generate a group of order 24, simply isomorphic with the symmetric group of degree 4. Thus, for instance, one such plane contains the nodes $(12), (13), (14), (23), (24), (34)$; and, since the effect of $p(ij)$ on the nodes is represented by the interchange of $i$ and $j$ in the symbols of the nodes, the operations of the group generated by the six projections effect the 24 permutations of 1, 2, 3, 4 in the symbols. Alternatively, since the projection from any node $C'$ on a $\kappa$-line $A'B'C'$ is expressible in the form $p(A')p(B')p(A')$, the projections from the six nodes $A, B, C, A', B', C'$ can all be expressed in terms of $p(A), p(B), p(A')$; and these projections satisfy the relations

$$[p(A)]^2 = [p(B)]^2 = [p(A')]^2 = [p(A)p(B)]^3 = [p(B)p(A')]^3 = [p(A)p(A')]^2 = 1,$$

which are defining relations for the symmetric group of degree 4 (Moore 1897; or Burnside 1911, p. 464).

Of the 24 operations of this group all but six belong to types already considered, for besides identity and six projections it is seen that there are three operations of type III arising from the three diagonals of the quadrilateral and eight operations of type IV, two arising from each of the four $\kappa$-lines. The remaining six operations may thus be identified with the six operations of period 4 in the symmetric group. Each of these operations can in fact be expressed, in a variety of ways, as the product of three projections. I shall content myself by showing, directly from the geometry of the figure, that they are in fact obtained by permuting $A, B, C$ in all possible ways in the product $p(A)p(B)p(C)$. It is an easy exercise in the use of elementary transformations to express such a product in terms of any other set of three projections from nodes whose joins are three of the four $\kappa$-lines in the plane. Observe, first, that $p(A)p(B)p(C)$ permutes the six nodes according to the permutation $(ACA'C')(BB')$, which are defining relations for the symmetric group of degree 4 (Moore 1897; or Burnside 1911, p. 464).
which shows that the six operations of this type are distinct. Further, by elementary transformations,

\[
[p(A)p(B)p(C)]^2 = p(A)p(B)p(C)p(A)p(B)p(C)
\]

\[
= p(A)p(B)p(C)p(B)p(A') = p(A)p(B)p(B')p(A')
\]

\[
= p(B)p(C')p(B') = p(B)p(B'),
\]

an operation of type III and period 2. Thus \(p(A)p(B)p(C)\) is of period 4. And the transforms of \(p(A)p(B)p(C)\) by \(p(A)\) and \(p(A')\) are easily seen to be \(p(B)p(C)p(A)\) and \(p(A)p(C)p(B)\). From this it follows that the six operations in question are all conjugate in \(G\). Hence, as \(G\) permutes the cross-planes transitively, the operations of this type form a single conjugate set in \(G\). Now the operations arising from different cross-planes are necessarily distinct. For suppose, if possible, that \(A_1 B_1 \ldots C_1'\) is a second cross-plane, the arrangement of the nodes on \(\kappa\)-lines being the same as in the plane \(AB\ldots C'\) (with the suffix 1 of course), and that

\[
p(A_1)p(B_1)p(C_1) = p(A)p(B)p(C).
\]

Then the squares of these operations, namely, \(p(B_1)p(B_1')\) and \(p(B)p(B')\), are equal. Hence the \(e\)-lines \(B_1 B_1'\) and \(BB'\) are the same, and \(B_1, B_1'\) coincide in some order with \(B\) and \(B'\). Since

\[
p(A_1)p(B_1)p(C_1) = p(A_1)p(A_1)p(B_1) = p(C_1)p(B_1)p(A_1'),
\]

it may be assumed, without loss of generality, that \(B_1 = B, B_1' = B'\). But then, since

\[
p(A_1)p(B_1)p(C_1) = p(A_1)p(A_1)p(B_1), \quad p(A)p(B)p(C) = p(A)p(A')p(B),
\]

it is seen that \(p(A_1)p(A_1') = p(A)p(A')\). Thus, \(A_1, A_1'\) coincide in some order with \(A, A'\) and the two planes must be identical. Hence, since each of the 540 \(c\)-planes gives rise to six operations of this kind, \(G\) contains

**VI. A conjugate set of 3240 operations of period 4, products of projections from three nodes of a cross-plane, whose squares are of type III.**

4-3. The situation arising in the case of a \(j\)-plane is not quite so simple. The nine nodes belonging to a \(j\)-plane lie by threes on 12 \(\kappa\)-lines, and the join of any two of these nodes passes through a third. It is familiar that the arrangement of such a set of nine points in triads of collinear points may be represented by the scheme

\[
\begin{array}{ccc}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9 \\
\end{array}
\]

in which 1, 2, ..., 9 denote the nine points, and the triads of collinear points are given by the rows, columns, diagonals and broken diagonals in the array. The nine nodes
of a Jacobian plane will be denoted by 1, 2, ..., 9 in the present section, and from the
array just given it is seen that the 12 \( \kappa \)-lines are

\[
123, \ 456, \ 789, \ 147, \ 258, \ 369, \ 159, \ 267, \ 348, \ 168, \ 249, \ 357.
\]

It will also be convenient to denote the projection from the node \( i \) by \( [i] \), the product
\( \{i\} \{j\} \) by \( \{ij\} \) and so on.

Any irreducible product of three projections from nodes in the plane has the
form \( \{ijk\} \), where \( i, j, k \) are three distinct nodes not lying on a \( \kappa \)-line. There are
\( 9 \times 8 \times 6 = 432 \) such products, and it will be proved that these give rise to 18 distinct
operations of \( G \), each product being expressible in 24 equivalent forms. In fact,
taking a typical product \( \{124\} \) and using the list of \( \kappa \)-lines just given, one can verify
without difficulty that the 24 products

\[
\{124\}, \ \{234\}, \ \{314\}, \ \{479\}, \ \{512\}, \ \{719\}, \ \{837\}, \ \{928\},
\{149\}, \ \{248\}, \ \{347\}, \ \{498\}, \ \{523\}, \ \{795\}, \ \{875\}, \ \{985\},
\{192\}, \ \{283\}, \ \{371\}, \ \{487\}, \ \{531\}, \ \{751\}, \ \{853\}, \ \{952\},
\]

are all equal. Moreover, \( \{124\} \) and its inverse \( \{421\} \) keep fixed the node 6 and inter­
change in pairs the nodes 1, 8; 2, 7; 3, 9; 4, 5 which lie on \( \kappa \)-lines through 6. Hence
the 432 products give rise to nine pairs of inverse operations such as \( \{124\} \) and \( \{421\} \),
each pair keeping fixed a definite one of the nine nodes and interchanging in pairs the
nodes lying on \( \kappa \)-lines through this. It remains to prove that \( \{124\} \) and \( \{421\} \) are
distinct. Now

\[
\{124\}^3 = \{124\} \{479\} \{928\} = \{12728\} = \{168\} = \{688\} = \{6\},
\]

so that \( \{124\}^3 \) is the projection \( \{6\} \). And if \( \{124\} = \{421\} \) then \( \{124\}^2 = 1 \), so that
\( \{124\} = \{124\}^3 = \{6\} \) and hence \( \{12\} = \{64\} \), in contradiction with the result proved
in § 3 that products of projections from two nodes of different \( \kappa \)-lines are distinct.
Thus there are precisely 18 operations such as \( \{ijk\} \) which are distinct, and each of
these is an operation of period 6.

Next observe that in the 24 different expressions of \( \{124\} \) as a product of pro­
jections \( \{ijk\} \) the last factor \( \{k\} \) takes every value except \( \{6\} \). Thus the only possible
irreducible product of four projections of the form \( \{124l\} \) is \( \{1246\} \). And since any
set of five or more of the nine nodes includes at least three which are collinear, no
product of more than four projections from the nodes in the plane can be irreducible.
Thus there remain for consideration 18 operations \( \{ijkl\} \), where \( \{ijk\} \) is one of the
18 operations of period 6 just considered and \( l \) is the node left invariant by \( \{ijk\} \).
It will be proved that in fact all these operations are equal either to \( \{1246\} \) or \( \{4216\} \).
Considerations of symmetry show that it suffices to consider a single pair of opera­
tions \( \{ijkl\}, \{jikl\} \), where \( l \neq 6 \).

Observe, first, that since \( \{124\}^3 = \{6\} \) and \( \{421\} = \{124\}^{-1} \), then

\[
\{1246\} = \{124\}^4 = \{421\}^2, \quad \{4216\} = \{124\}^2,
\]
so that $\{1246\}$ and $\{4216\}$ are inverse operations, each of period 3. It is thus merely necessary to show that the squares of another typical pair of inverse operations $\{ijk\}, \{kji\}$ coincide with $\{1246\}$ and $\{4216\}$. Taking, for instance, the operations $\{125\}, \{521\}$, which leave the node 4 unchanged, it is seen that

$$\{125\}^2 = \{125125\} = \{125598\} = \{1298\} = \{1428\} = \{4728\} = \{4268\} = \{4216\},$$

$$\{521\}^2 = \{125\}^{-2} = \{4216\}^{-1} = \{1246\}.$$ 

There are thus just two operations of this type, and the square of any one of the 18 operations of period 6 coincides with one of them. Hence the projections from the nodes of a Jacobian plane generate a group of order 54, comprising identity, nine projections, 24 operations of type IV and the 18 operations of period 6 and the two operations of period 3 just obtained.

It may be remarked, in passing, that the operation $S$ of period 3 which is one of Burkhardt’s generating operations, is of the last type discussed, and several expressions for it as a product of four projections are given on p. 77 of Baker’s tract.

Each of the operations $\{1246\}, \{4216\}$ leaves invariant every node in the Jacobian plane. It can leave no other node unaltered. For considerations of incidence show that any operation of $G$ which keeps fixed all the nodes in a $j$-plane and a node $P$ not lying in this plane keeps fixed each node in the Steiner prime joining the $j$-plane to $P$, so that it keeps fixed each $c$-plane in this prime and hence the remaining 27 nodes, whose polar primes intersect the Steiner prime in these $c$-planes; such an operation of $G$ must therefore be the identity. It follows that the two operations of period 3 derived from each of the 40 $j$-planes give 80 distinct operations. And hence the operations of period 6 derived from different $j$-planes, whose squares are operations of period 3 of the type just mentioned, must also be distinct.

It will now be proved that the operations of each of the two types mentioned form two conjugate sets in $G$. For this observe, first, that as $G$ is transitive on the $j$-planes every operation (of either type) is conjugate to an operation of the same type associated with any given $j$-plane. And since the group generated by the projections $\{i\}$ permutes the nine nodes $i$ transitively, any operation of period 6 in the plane is conjugate to one or other of the operations $\{124\}, \{421\}$ which leave the node 6 unchanged. Now, if there exists an operation $T$ of $G$ such that $T\{124\} T^{-1} = \{421\}$, then, also, $T\{124\}^2 T^{-1} = [T\{124\} T^{-1}]^2 = \{421\}^2$. Thus, if $\{124\}$ and $\{421\}$ are conjugate in $G$, so also are $\{1246\}$ and $\{4216\}$. It will now be shown that this is not the case.

In fact, if there exists an operation $T$ in $G$ such that $T\{1246\} T^{-1} = \{4216\}$, then the nine nodes $i' = T(i)$ lie in a Jacobian plane, and the set of four nodes $1'2'4'6'$ is projective with 1246 and such that $\{1'2'4'6'\} = \{4216\}$. Since distinct Jacobian planes give rise to distinct operations it follows that the plane of the nodes $i'$ coincides with that of the nodes $i$. Thus there is a projectivity in the Jacobian plane which keeps the set of nine nodes unaltered and transforms $i$ into $i'$. Now the group of projectivities which leave the set of nine nodes unaltered is known to be of order 216, and the group of automorphisms of the set of nodes is of order 432, and consists of
216 collineations and 216 anticollineations. There are precisely 432 ordered sets of nodes \(ijk\) which are not collinear, and there exists a unique automorphism of the nine nodes which transforms any one of these sets into any other. Each such triad of nodes determines uniquely a fourth node \(l\) such that \(\{ijkl\}\) leaves each node unaltered. Thus, if \(\{ijkl\}\) and \(\{i'j'k'l'\}\) are two expressions of this type, there is a unique automorphism which transforms \(i, j, k, l\) respectively into \(i', j', k', l'\). It is not difficult to see that any two operations of this type which can be transformed into one another by elementary transformations must be such that the corresponding automorphism is a collineation. Hence, if \(\{1246\}\) and \(\{4216\}\) are conjugate, there is a projectivity \(T_1\) leaving the set of nodes invariant as a whole which transforms \(1', 2', 4', 6'\) into \(4, 2, 1, 6\) respectively. Hence, if \(T_2\) is the projectivity induced among the nine nodes by \(T, T_1T_2\) is a projectivity of the set of nodes in which \(1, 2, 4, 6\) correspond respectively to \(4, 2, 1, 6\). In this projectivity the node 3 on the line 12 must correspond to the node 9 on the line 42. Hence the pencils \(6(1243)\) and \(6(4219) = 6(4213)\) are projective and consequently harmonic, whereas they are known to be equiunharmonic. Hence \(\{1246\}\) and \(\{4216\}\) are not conjugate in \(G\) (although they are conjugate in \(G^*\), being transformed into each other by an anticollineation). So, from the Jacobian planes, one obtains

VII. 80 operations of period 3, products of four projections, forming two conjugate sets of 40 operations each,

and

VIII. 720 operations of period 6, products of three projections, forming two conjugate sets of 360 operations each, whose squares are of type VII and whose cubes are of type II.

5. Products of projections from sets of nodes in a prime

5·1. Now consider the products of projections from sets of nodes lying in a prime, and examine, in turn the four types of prime described in § 1. The possibilities presented by an \(x\)-prime are soon disposed of. For an \(x\)-prime is the join of a cross-plane to the pole of the Jordan prime which contains it. If, as in § 4·2, the six nodes of the cross-plane are denoted by \(A, B, C, A', B', C'\) and the pole of the Jordan prime containing it by \(O\), the projection \(p(O)\) is easily seen to commute with the projections from the other six nodes. Hence the group generated by the projections from the nodes of the prime is of order 48, and is the direct product of the symmetric group of degree 4 generated by the projections from the nodes in the cross-plane and the cyclic group of order 2 generated by \(p(O)\). And the only operations of this group which do not clearly belong to types already obtained are the products of \(p(O)\) with the six operations of period 4 (type VI) belonging to the cross-plane. But, in the notation of § 4·2,

\[ p(A)p(B)p(C)p(O) = p(B)p(C)p(O), \]

and this product is reducible since the plane \(OCC'\) is an \(f\)-plane. Thus no new type of operation arises. Incidentally, since \(G\) permutes the 540 \(x\)-primes transitively, the group of order 48 just mentioned is just the subgroup of \(G\) which keeps the
x-prime fixed. It is easily seen that the 48 operations of this group comprise, besides identity, 7 operations of type II, 12 of type III, 8 of type IV, 8 of type V, and 12 of type VI.

The operations derived from the nodes of an n-prime are likewise easily discussed. For a typical n-prime is that containing the ten nodes (ij) in which neither i nor j is 6, and since the effect of $p(ij)$ on the nodes is represented by the interchange of i and j in the symbols of these nodes it follows that the projections from the ten nodes (ij) generate a group of order 120, simply isomorphic with the symmetric group of degree 5. And, since there are 216 n-primes permuted transitively by the operations of G, this group is simply the subgroup of G which leaves the n-prime invariant. It is easy to see that the only operations of this subgroup which are not of a type already considered are the 24 operations of period 5 which permute the symbols 1, 2, 3, 4, 5 in some cyclic order. And since any cyclic permutation (ijklm) is equivalent to the product of four transpositions—for example, to the product $p(ml)p(mk)p(mj)p(mi)$—it will be seen that each of these operations is expressible as the product of four projections, the product $p(ml)p(mk)p(mj)p(mi)$ being a typical one. These 24 operations are conjugate in the symmetric group of degree 5, and hence are conjugate in G, which contains this symmetric group as a subgroup. Hence, since G permutes the 216 n-primes transitively, all the operations of this type form a complete set of conjugates in G. Each of these operations permutes the ten nodes of the n-prime from which it arises in two cycles of 5, the five nodes in any cycle not all lying in a plane. But it is easily seen, from the incidence relations of § 1, that any two n-primes have at least one of the nodes in common. Whence it follows that no two operations arising from different n-primes can be identical. Thus G contains

IX. A conjugate set of 5184 operations of period 5, each the product of four projections.

The 120 operations which keep the n-prime fixed comprise identity, ten operations of type II, 15 of type III, 20 of type IV, 20 of type V, 30 of type VI, and 24 of type IX.

5-2. Consider next a Jordan prime II, whose pole is denoted by O. Then II contains 12 nodes forming the vertices of three desmic tetrahedra, any one of which, together with O, forms a Jordan pentahedron. These tetrahedra will be denoted by $\sigma_i$ ($i = 1, 2, 3$) and the vertices of $\sigma_i$ by $A_i, B_i, C_i, D_i$; and symbols such as $P_i, Q_i, ...$ will be used to refer to unspecified vertices of $\sigma_i$. The line joining any vertex of $\sigma_1$ to any vertex of $\sigma_2$ contains a vertex of $\sigma_3$, and the notation may be chosen in such a way that the vertex of $\sigma_3$ is determined by the scheme

<table>
<thead>
<tr>
<th></th>
<th>$A_2$</th>
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<td>$A_1$</td>
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<tr>
<td>$B_1$</td>
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<td>$A_3$</td>
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<tr>
<td>$C_1$</td>
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<td>$C_3$</td>
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<td>$A_3$</td>
</tr>
</tbody>
</table>
Any product of projections from a set of non-coplanar nodes involves four factors at least, and so, if the nodes lie in \( \Pi \), must contain at least two projections from nodes of the same tetrahedron. If it contains three such projections it is reducible. Thus any irreducible product of projections from non-coplanar nodes in \( \Pi \) can be reduced by elementary transformations to the form \( p(P_i) p(Q_i) T \), where \( T \) is a product of two or more projections from nodes of the tetrahedra \( \sigma_j, \sigma_k \) distinct from \( \sigma_i \). If \( T \) contains as factors two projections \( p(R_j), p(S_k) \) from nodes of different tetrahedra, then since the \( \kappa \)-line \( R_jS_k \) contains a node belonging to \( \sigma_i \) one can transform \( p(P_i) p(Q_i) T \) into a form involving three projections from nodes of \( \sigma_i \), and this product is accordingly reducible. It follows at once that every product of five or more projections from nodes of \( \Pi \) is reducible, and that any irreducible product of four projections must be expressible in the form

\[
p(P_i) p(Q_i) p(R_j) p(S_j),
\]

where \( P_i Q_i, R_j S_j \) are non-intersecting edges of distinct tetrahedra \( \sigma_i, \sigma_j \). Since the edges of the tetrahedra are \( e \)-lines the projections from \( P_i, Q_i \) (or \( R_j, S_j \)) commute. Thus there are essentially \( 18 \cdot 8 = 144 \) products of this type to consider. It will be shown that these determine 12 distinct operations.

Consider, for instance, the typical product \( p(A_1) p(B_1) p(A_2) p(C_2) \). Then, by elementary transformation,

\[
p(A_1) p(B_1) p(A_2) p(C_2) = p(A_3) p(A_3) p(B_3) p(C_3),
\]

and the latter product can be transformed, in two ways, to give

\[
p(A_2) p(C_2) p(C_1) p(D_1) \quad \text{and} \quad p(A_3) p(B_3) p(B_2) p(C_2).
\]

By means of transformations of this type it is seen that in any product

\[
p(P_i) p(Q_i) p(R_j) p(S_j)
\]

of the type considered the two tetrahedra \( \sigma_i, \sigma_j \) to which \( P_i, Q_i, R_j S_j \) belong may be assigned in advance. Moreover,

\[
p(A_1) p(B_1) p(A_2) p(C_2) = p(A_1) p(B_1) p(O) p(O) p(A_2) p(C_2) = p(C_1) p(D_1) p(B_2) p(D_2),
\]

since \( O \) forms a Jordan pentahedron with each of the tetrahedra \( \sigma_i \), so that the same operation arises if each of the edges \( P_i Q_i, R_j S_j \) is replaced by the opposite edge of the tetrahedron to which it belongs. There are thus at most 12 distinct operations of this type, all expressible in the form \( p(A_1) p(P_1) p(R_2) p(S_2) \), where \( P_1 \) is a vertex of \( \sigma_1 \) distinct from \( A_1 \) and \( R_2 S_2 \) is one of the four edges of \( \sigma_2 \) which do not meet \( A_1 P_1 \). These operations are easily seen to be distinct, for \( p(P_1) p(R_2) p(S_2) \) is an operation of type VI, associated with the cross-plane \( P_1 R_2 S_2 \) (which does not contain \( A_1 \)), interchanging the two nodes of this cross-plane which belong to \( \sigma_1 \), and two distinct
(inverse) operations of this kind arise from each of the six cross-planes in \( \Pi \) which do not contain \( A_1 \). Further, since

\[
[p(A_1) p(B_1) p(A_2) p(C_2)]^2 = p(A_1) p(B_1) p(A_2) p(C_2) \cdot p(A_1) p(B_1) p(A_2) p(C_2) = p(A_1) p(B_1) p(C_1) p(D_1) = p(O),
\]
each of these operations is of period 4, and operations arising from different Jordan primes have different squares and so are distinct. By using identities such as

\[
p(A_2) p(A_1) p(B_1) p(A_2) p(C_2) p(A_2) = p(A_1) p(B_1) p(B_2) p(C_2),
\]
\[
p(B_1) p(A_1) p(B_1) p(A_2) p(C_2) p(B_1) = p(A_1) p(D_1) p(A_2) p(C_2),
\]
it is easy to see that the 12 operations derived from \( \Pi \) are all conjugate. Hence, as \( G \) is transitive on the Jordan primes, the operations of this type in \( G \) form a single conjugate set, and hence \( G \) contains

\[ X. \text{ A conjugate set of 540 operations of period 4, products of four projections, whose squares are of type II.} \]

The group generated by the projections from the nodes in \( \Pi \) may now be determined. Since \( p(O) \) belongs to the group it is necessary to consider not merely the operations of various types derived from nodes in \( \Pi \) but also the products of these with \( p(O) \), when these products are irreducible. It is not difficult to verify that this group is of order 192, and that its operations consist of (i) identity, (ii) the projection \( p(O) \), (iii) the 12 projections from nodes in \( \Pi \), (iv) the 12 products of these with \( p(O) \), which are of type III, (v) the 18 products of projections from two nodes on an edge of one of the tetrahedra in \( \Pi \) (type III), the product of such an operation by \( p(O) \) being the corresponding operation derived from the opposite edge of the tetrahedron, (vi) 32 operations of type IV, two arising from each of the 16 \( \kappa \)-lines in \( \Pi \), (vii) the 32 products of these with \( p(O) \), which are of type V, (viii) 72 operations of type VI, six arising from each of the 12 \( c \)-planes in \( \Pi \), the product of any one of which with \( p(O) \) is another operation of the same set, (ix) the 12 operations of type \( X \) arising from \( \Pi \), the product of any one of which with \( p(O) \) is the inverse operation. This group has the cyclic group generated by \( p(O) \) as a self-conjugate subgroup, the two operations of which leave fixed each node in \( \Pi \), and the nodes of \( \Pi \) are permuted by the operations of the group of order 192 according to the 96 operations of the factor group of this cyclic group. The group of order 192 is, as will be seen presently, a subgroup of the group of order 576 consisting of all the operations of \( G \) which leave \( \Pi \) and its pole \( O \) invariant.

5-3. Next consider the products of projections from nodes lying in a Steiner prime \( \Sigma \). Such a prime contains 18 nodes lying by threes on the edges of a tetrahedron whose faces are Jacobian planes and whose edges form three pairs of polar \( \kappa \)-lines. Baker has given, on p. 79 of his tract, explicit equations for the projections from the 18 nodes of one typical Steiner prime. It will be found convenient to denote the nodes of \( \Sigma \) by the symbols used by Baker to denote these projections. Accordingly, the nodes will be denoted by \( X, Y, Z, P, Q, R, U, V, W, X', Y', Z', P', Q', R', U', V', W' \),
the notation being such that \(XYZ, X'Y'Z'\); \(PQR, P'Q'R'\); \(U VW, U'V'W'\) are the three pairs \(x, x'\); \(p, p'\); \(u, u'\) of opposite edges of the tetrahedron. Reference to Baker's tract shows that the arrangements of the nodes in the four Jacobian planes are given by the scheme

\[
\begin{align*}
XYZ & \quad X'Y'Z' & \quad X'Y'Z' \\
PRQ & \quad P'R'Q' & \quad P'QR & \quad P'Q'R' \\
WVU & \quad W'V'U' & \quad V'U'W' & \quad VUW
\end{align*}
\]

where, in each set of nine nodes, the arrangement on \(\kappa\)-lines is determined, as in §4·3, by the rule that three nodes on a \(\kappa\)-line occur either all in the same row or column or else each in a different row and column.

The examination of sets of nodes of \(\Sigma\) which give rise to irreducible products of projections may be carried out by methods similar to these used in the discussion of the Jordan prime, but this procedure is somewhat laborious and not particularly interesting, although it was the method by which the types of operation obtained in this section were originally derived. I shall proceed here in a different manner, first examining certain products which prove to yield new types of operation, and then showing that all irreducible types have been discovered by proving that all the operations of the group generated by the projections from nodes of \(\Sigma\) (which is necessarily a subgroup of the group of operations of \(G\) which keep \(\Sigma\) fixed, and proves in fact to coincide with this group) are accounted for by operations of types already determined.

The first type of product considered is a product of four projections of the form \(T_1T_2\), where \(T_1\) is the product of projections from two nodes on an edge of the tetrahedron and \(T_2\) is the product of projections from two nodes on the opposite edge. Since opposite edges of the tetrahedron are polar \(\kappa\)-lines \(T_1\) and \(T_2\) commute. Hence, as each \(\kappa\)-line gives rise to two distinct products of two projections, there are 12 products of this type to consider (not regarding \(TXT_2\) and \(T_2TX\) as distinct). As a matter of fact these 12 products give rise to six distinct operations, each being expressible in two essentially different ways. In fact, as is stated by Baker, there are identities of the form

\[
p(Y)p(Z)p(Y')p(Z') = p(V)p(W)p(V')p(W'),
\]

which may be proved by noticing that

\[
p(Y)p(Z)p(Y')p(Z') = p(Y)p(Z)p(P)p(P)p(Y')p(Z')
= p(P)p(U)p(V).p(W').p(U')p(P)
= p(U)p(V)p(R).p(R)p(W')p(U')
= p(U)p(V)p(W')p(U') = p(V)p(W)p(V')p(W').
\]

The typical operation \(p(Y)p(Z)p(Y')p(Z')\) is easily seen to keep fixed each node on \(p, p'\) and to permute cyclically the nodes on each of the other edges of the tetrahedron. Whence it is easily seen that there are six distinct operations of this type associated with \(\Sigma\), and that these can be transformed into one another by means of
projections from nodes lying in $\Sigma$, so that they are all conjugate. Moreover, no $\kappa$-line of $\Sigma$ other than the edges of the tetrahedron, and no $c$-plane lying in $\Sigma$ (which contains one node lying on each edge of the tetrahedron) is left unaltered by this operation. Hence different Steiner primes must give rise to distinct operations. Since $G$ permutes the Steiner primes transitively, all these operations are conjugate in $G$. Since $T_1$ and $T_2$ are commuting operations of period 3, $T_1 T_2$ has period 3. Hence $G$ contains

XI. A conjugate set of 240 operations of period 3, products of four projections, each expressible in two ways as a product of two operations of type III arising from a pair of polar $\kappa$-lines.

The next type of product to be considered is of the form $T_1 \cdot p(A)$, where $T_1$ is an operation of type VIII, product of three projections from nodes in a Jacobian plane in $\Sigma$ leaving a node $B$ of this plane fixed, and $A$ is a node lying on the edge of the tetrahedron opposite to that which contains $B$. There are 18 operations such as $T_1$ associated with any one of the Jacobian planes, and three positions possible for $A$ corresponding to any operation $T$. Thus each Jacobian plane gives rise to 54 products of this type. It will be proved that the same set of operations arises from each of the four Jacobian planes in $\Sigma$, and that the 54 products which arise are all distinct.

Consider the typical operation $T = p(X) p(Y) p(U) p(U')$, where $p(X) p(Y) p(U)$ is an operation of type VIII associated with the Jacobian plane $xpu$ leaving the node $V$ fixed, and $U'$ lies on the polar $\kappa$-line of the edge $u$ of the tetrahedron which contains $V$. Then

\[
\begin{align*}
p(X) p(Y) p(U) p(U') &= p(X) p(Y) p(U') p(U), \\
p(X) p(Y) p(U) p(U') &= p(U') p(R') p(P') p(U) = p(R') p(P') p(W') p(U) \\
&= p(R') p(P') p(U) p(W'), \\
p(X) p(Y) p(U) p(U') &= p(U) p(R) p(P) p(U') = p(R) p(P) p(W) p(U') \\
&= p(R) p(P) p(U') p(W),
\end{align*}
\]

and

\[
\begin{align*}
p(X) p(Y) p(U'), p(R') p(P') p(U), p(R) p(P) p(U')
\end{align*}
\]

are operations of type VIIII associated respectively with the Jacobian planes $xp'u', x'p'u, x'pu'$, leaving unaltered, respectively, the nodes $V', V, V'$. Thus the operation $T$ arises equally from any one of the Jacobian planes. The operation $T$ is easily seen to permute the nodes in $\Sigma$ according to the permutation

\[
(XX'YZ'ZY')(PR'QP'RQ')(UW)(U'W'),
\]

so that it leaves $V$ and $V'$ unaltered, interchanges the remaining nodes on the edges $u,u'$, and interchanges the other two pairs of opposite edges of the tetrahedron, permuting the six nodes on the edges of each pair in a cycle of 6. There are two distinct operations of this type (inverse to each other) keeping fixed any one of the 27 pairs of nodes such as $V, V'$ lying on opposite edges of the tetrahedron. Thus the 54 operations so arising are distinct. Further, $T$ interchanges the Jacobian planes
(xpu, x'p'u) and (xp'u, x'pu') and permutes the three cross-planes of \( \Sigma \) which contain \( V \) and \( V' \), namely, the planes containing

\[ VV'XZ'QQ', \quad VV'ZX'PP', \quad VV'Y Y' RR', \]

in cyclic order. Whence it is easily seen that two distinct Steiner primes cannot give rise to the same operation.

By elementary transformations it is found that

\[
T^2 = p(X)p(Y)p(U)p(U').p(X)p(Y)p(U)p(U') = p(X)p(Y)p(U)p(U'),
\]

an operation of period 3 and type XI, and that

\[
T^3 = p(X)p(Y)p(X')p(Z')p(X)p(Y)p(U)p(U') = [p(X)p(Y)]^2p(X')p(Z')p(U)p(U') = p(Y)p(X)p(U)p(U')p(X)p(Y) = p(V)p(V'),
\]

an operation of period 2 and type III. Thus \( T \) is of period 6.

It is easy to see that the subgroup of \( G \) which keeps \( E \) fixed contains operations which transform any pair of opposite edges of the tetrahedron into any other pair, and operations (products of projections from suitably chosen nodes on \( u \) and \( u' \)) which keep \( u \) and \( u' \) unchanged and transform \( V, V' \) into any other assigned pair of nodes on these edges. Thus every one of the 54 operations \( T \) derived from \( \Sigma \) is conjugate to one of the two operations \( T, T^{-1} \) which keep \( V \) and \( V' \) fixed. And \( T \) and \( T^{-1} \) are conjugate, for

\[
T = p(X)p(Y)p(U)p(U'), \quad T^{-1} = p(U')p(U)p(Y)p(X), \quad T = p(Y)p(X)p(U)p(U')p(X)p(Y),
\]

and

\[
[p(Y)p(Y')] T[p(Y)p(Y')]^{-1} = p(Y)p(Y')p(X)p(Y)p(U)p(U')p(Y)p(Y)
\]

\[
= p(Y)p(X)p(Q)p(Q')p(Y)p(Y)
\]

\[
= p(Y)p(X)p(W)p(W) = T^{-1}.
\]

Thus all the 54 operations derived from \( \Sigma \) are conjugate. And since \( G \) is transitive on the Steiner primes the operations of this type corresponding to different primes are all conjugate in \( G \). Thus \( G \) contains

\[ \text{XII. A conjugate set of 2160 operations of period 6, products of four projections, whose squares are of type XI and whose cubes are of type III.} \]

The next type of product that arises can be put in the form \( T_1 \cdot p(A) \) where, again, \( T_1 \) is an operation of type VIII associated with one of the Jacobian planes in \( \Sigma \) leaving a node \( B \) in this plane invariant, and \( A \) is a node in \( \Sigma \), not lying in the Jacobian plane, such that \( A \) and \( B \) lie on intersecting edges of the tetrahedron. For each operation \( T_1 \) there are thus six possible positions for \( A \). Hence there are, in all,
4.18.6 = 432 products of this nature to be considered. I shall prove that these lead to 144 distinct operations.

A typical operation of this type is \( T = p(X)p(Y)p(U)p(X') \). By elementary transformations it is seen that

\[
p(X)p(Y)p(U)p(X') = p(X)p(Y)p(U)p(X') = p(U')p(V')p(U)p(X')
\]

\[
= p(U')p(V')p(U)p(X') = p(Y)p(Q)p(R)p(Q')
\]

\[
= p(Q)p(R)p(X')p(Q').
\]

Thus \( T \) is expressible in the form \( T_1.p(A) \) in three ways, in which the operation \( T_1 \) belongs to the respective Jacobian planes \( xpu, xp'u', x'pu' \). Hence the 432 products give rise to at most 144 distinct operations. The operation \( T \) permutes the 18 nodes lying in \( \Sigma \) in the cycles

\[
(XQV'ZPU'YRW')(X'Q'U)(Y'R'W)(Z'P'V).
\]

From this it is easily seen (i) that no cross-plane lying in \( \Sigma \) is unaltered by \( T \), (ii) that the three Jacobian planes in \( \Sigma \) other than \( x'p'u' \) are permuted cyclically, (iii) that the Jacobian plane \( x'p'u' \) is left unchanged, but that the three triads of \( \kappa \)-lines

\[
(X'Q'W, Y'R'V, Z'P'U), \ (Q'UY', R'WZ', P'VX'), \ (UX'R', WY'P', VZ'Q'),
\]

which are the edges, lying in this plane, of the tetrahedra of Jacobian planes in the three Steiner primes other than \( \Sigma \) which contain the plane \( x'p'u' \), are permuted cyclically. From these properties it is easy to see that distinct Steiner primes give distinct operations.

It will now be proved that there are just 144 distinct operations of this type derived from \( \Sigma \). In the first place it is to be observed that

\[
T^3 = p(X)p(Y)p(U)p(X').p(X)p(Y)p(U)p(X').p(X)p(Y)p(U)p(X')
\]

\[
= [p(X)p(Y)]^3.p(V)p(X')p(U)p(X').p(X)p(Y)p(U)p(X')
\]

\[
= p(Y)p(X).p(X)p(Y)p(W)p(P')p(U)p(X')
\]

\[
= p(W)p(P')p(U)p(X'),
\]

which is easily recognized as an operation of type VII and period 3 associated with the Jacobian plane \( x'p'u' \). Thus, since \( T \) contains a cycle of period 9, \( T \) is an operation of period 9. The three operations \( T, T^4, T^7 \) effect the same permutation of the nine nodes in the plane \( x'p'u' \), this permutation consisting of three cycles each of three non-collinear nodes. It is easy to see that the three operations \( p(X)p(Y)p(U)p(A) \), where \( A = X', Y' \) or \( Z' \), the transforms of these operations by \( p(X') \), and the squares of these six operations (which are all of the same type as \( T \)), effect 12 different permutations of the nine nodes in \( x'p'u' \). Thus these 12 operations, together with their fourth and seventh powers (which may be verified to be of the same type)

\[\dagger\] In fact, \( T^2 = p(Y)p(X)p(V)p(R') \).

\[\ddagger\] In fact, \( T^4 = p(X)p(Y)p(V)p(Z'), T^7 = p(X)p(Y)p(W)p(Y') \).
give 36 operations of this type, all distinct, which leave the plane $x'p'u$ unchanged. Thus, from the four planes $xpu, xp'u$, $x'pu', x'p'u$ we get 144 distinct operations.

These operations cannot all be conjugate in $G$, for their cubes, which are of type VII, belong to two distinct conjugate sets. It will be proved, in fact, that they form two conjugate sets in $G$. In the first place, since

$$p(X)p(Y)p(U) = p(U)p(R)p(P) = p(R)p(P)p(W),$$

the operations $p(X)p(Y)p(U)p(X')$ and $p(R)p(P)p(W)p(X')$ are identical. It follows that every operation $T_1, p(A)$ of the type considered can be expressed as a product $p(X_1)p(Y_1)p(U_1)p(X'_1)$ of projections from four nodes $X_1, Y_1, U_1, X'_1$ lying in a Steiner prime $\Sigma_1$ with the property that $X_1Y_1$ is an edge of the tetrahedron formed by the Jacobian planes in $\Sigma_1$, $U_1$ lies in one of these planes through $X_1Y_1$, and $X'_1$ lies on the polar $\kappa$-line of $X_1Y_1$. Now there is a unique automorphism of the set of nodes in which $X, Y, U, X'$, correspond respectively to $X_1, Y_1, U_1, X'_1$; for incidence relations determine uniquely (i) the node corresponding to the third node $Z$ on the $\kappa$-line $XY$, (ii) the nodes corresponding to the remaining nodes in $\Sigma$, (iv) the nodes corresponding to the remaining 27 nodes, which are the poles of the Jordan primes containing the various cross-planes in $\Sigma$. Hence any two products of the type $T_1, p(A)$ are conjugate in the group $G^*$ of automorphisms of the set of nodes. Since $G$ is a subgroup of index 2 in $G^*$ these operations form at most two conjugate sets in $G$. But it has been seen that they cannot all belong to the same conjugate set. Thus they form exactly two conjugate sets, and hence $G$ contains

XIII. 5760 operations of period 9, products of four projections, forming two conjugate sets of 2880 operations each, whose cubes are of type VII.

It is not difficult to prove that any irreducible product of four projections from nodes of $\Sigma$ which do not lie in a plane can be reduced by elementary transformations to one of the three types XI, XII, XIII. Any other irreducible product thus contains at least five factors. There is, in fact, just one further type of product which is irreducible, namely a product of five projections expressible in the form $T_1, p(A)$, where $T_1$ is an operation of type VII, the product of projections from four nodes in a Jacobian plane lying in $\Sigma$, which leaves every node in this plane invariant, and $A$ is a node in $\Sigma$ which does not lie in this plane. Since each Jacobian plane determines two such operations $T_1$ and there are nine nodes of $\Sigma$ not lying in any given plane, there are 72 products of this type to be considered. In fact these give rise to 36 distinct operations.

Consider the typical product $T = p(X)p(Y)p(P)p(Q)p(X')$. It is easily seen that this operation leaves invariant the four nodes $X, Y, Z, Z'$, interchanges $X'$ and $Y'$, and permutes the remaining nodes in $\Sigma$ in two cycles of six, namely,

$$(PW'QU'RV')(P'R'V'UQ'W).$$

Thus $T$ leaves invariant the two Jacobian planes $x'p'u, x'p'u$ which pass through $x'$ and interchanges the two Jacobian planes $xpu, xp'u'$ which pass through the line $x$.
containing three invariant nodes. Considerations of a nature similar to those previously employed show that distinct Steiner primes must give rise to distinct operations.

Since \( T = p(X)p(Y)p(P)p(Q)p(X') = p(X)p(Y)p(X')p(V')p(W') \)
\[ = p(X)p(Y)p(V')p(W')p(Y'), \]
and \( p(X)p(Y)p(V')p(W') \) is an operation of type VII associated with the Jacobian plane \( xp'u' \), each operation of this type is expressible in at least two ways in the form \( T_1.p(A) \). On the other hand, the two operations \( T \) and \( T^{-1} \) (which are distinct since they effect distinct permutations of the twelve nodes on the edges \( p, p', u, u' \) of the tetrahedron) keep fixed \( X, Y, Z, Z' \), that is, they keep fixed \( Z' \) and each node of the polar \( \kappa \)-line of the edge of the tetrahedron on which \( Z' \) lies, and by symmetry a similar pair of operations is associated with each of the 18 nodes in \( \Sigma \). There are thus exactly 36 distinct operations of this type derived from each Steiner prime.

Since
\[ p(X)p(Y)p(P)p(Q) = p(R)p(P)p(X)p(Y), \]
it follows that
\[ T^2 = p(R)p(P)p(X)p(Y)p(X') .p(X)p(Y)p(P)p(Q)p(X') \]
\[ = p(R)p(P)[p(X)p(Y)]^2 .p(X')p(P)p(Q)p(X') \]
\[ = p(R)p(P)p(Y)p(X)p(V')p(W') \]
\[ = p(R)p(P)p(Y)p(Q')p(P')p(X) \]
\[ = p(R)p(P)p(Q')p(P')p(X)p(X) = p(R)p(P)p(Q')p(P'), \]
an operation of type XI and period 3, and
\[ T^3 = p(X)p(Y)p(P)p(Q)p(X') .p(R)p(P)p(Q')p(P') \]
\[ = p(X)p(Y)p(P)p(Q)p(R)p(P).p(Z')p(Q')p(P') \]
\[ = p(X)p(Y)p(Q)p(P).p(V)p(U)p(Z') \]
\[ = p(X)p(Q)p(X)p(V)p(Z') \]
\[ = p(Z'), \]
so that \( T \) is of period 6 (and incidentally does not belong to a type already considered).

It has been seen that \( T \) can be expressed in the form \( T_1.p(A) \), where \( T_1 \) is an operation of type VII in just two ways, the two operations \( T_1 \) being
\[ p(X)p(Y)p(P)p(Q) \quad \text{and} \quad p(X)p(Y)p(V')p(W'), \]
and the corresponding projections \( p(A) \) being \( p(X') \) and \( p(Y') \). Since
\[ p(Z').p(X').p(Z') = p(Y') \]
and
\[ p(Z').p(X)p(Y)p(P)p(Q).p(Z') = p(X)p(Y)p(U')p(V') = p(X)p(Y)p(V')p(W'), \]
it is seen that the two operations $T_1$ which give rise to $T$, and the two corresponding projections $p(A)$, are transformed into each other by the same operation of $G$. Hence if $T'_1, p(A')$ is any other operation of this type which is conjugate to $T$, then $T'_1$ must be conjugate to $T_1$. Since the operations of type VII form two conjugate sets in $G$ there must therefore be at least two conjugate sets of operations of the type under consideration. But, by reasoning similar to that used in discussing the operations of type XIII, it is easy to see that any two operations of this type are conjugate in $G^*$, the group of automorphisms of the set of nodes. Hence the number of conjugate sets is seen to be exactly two, and hence $G$ contains

XIV. 1440 operations of period 6, products of five projections, forming two conjugate sets of 720 operations each, whose squares are of type VII and whose cubes are projections.

Every product of projections from a set of nodes lying in $\Sigma$ belongs to a type already considered. For the projection from any node in $\Sigma$ leaves $\Sigma$ unchanged, so that the group generated by the projections from the 18 nodes in $\Sigma$ either coincides with, or is a subgroup of, the group of order 648 consisting of all the operations of $G$ leaving $\Sigma$ invariant. In fact, it coincides with this group of order 648, and its operations consist of (i) identity, (ii) 18 projections from nodes in $\Sigma$, (iii) 27 operations of type III arising from $e$-lines in $\Sigma$, (iv) 84 operations of type IV arising from $\kappa$-lines, (v) 36 operations of type V arising from $d$-planes, (vi) 162 operations of type VI arising from $c$-planes, (vii) 8 operations of type VII and 72 of type VIII arising from $j$-planes, and (viii) 6 operations of type XI, 54 of type XII, 144 of type XIII, and 36 of type XIV; making 648 operations in all. Thus every product of projections from a set of nodes in $\Sigma$ has been accounted for.

6. THE REMAINING OPERATIONS OF $G$, AND THE SUBGROUP LEAVING A NODE INVARIANT

Now return to the discussion of the subgroup of $G$, which will be denoted by $G_5$ (in accordance with the notation of § 7 below), leaving invariant a node $O$ and the Jordan prime $II$ which is the polar of $O$. This group is of order 576. It was proved, in § 5.2, that the projections from the nodes in $II$ generated a subgroup of $G_5$, of order 192, and the operations of this subgroup were enumerated in nine sets (i) to (ix). The other operations of known types which belong to $G_5$ are easily seen to be: (x) 16 operations of type VII, two associated with each Jacobian plane through $O$; (xi) 16 operations of type VIII, two associated with each Jacobian plane through $O$, which are the products of $p(O)$ with the operations of the previous set; (xii) 32 operations of type XI; (xiii) 96 operations of type XII; (xiv) 128 operations of type XIV, falling into two sets, of 32 and 96 operations respectively, which are the products of $p(O)$ with the operations of the two preceding sets. In each of the last three cases, the number of operations in the set is $nm/45$, where $n$ is the total number of operations of the type considered and $m$ is the number of nodes left unchanged by an operation
of this type. The division of the 128 operations of the set (xiv) into sets of 32 and 96 arises from the fact that of the four nodes left invariant by an operation of type XIV a particular set of three lies on a $\kappa$-line.

These fourteen sets account for 480 operations of $G_5$. It will be proved that the remaining 96 operations belong to a new type, and that this type, together with those obtained previously, exhausts the operations of $G$.

Since $p(O)$ leaves each node of $\Pi$ invariant (and is the only operation of $G$ other than identity with this property) the operations of $G_5$ permute the 12 nodes of $\Pi$ according to a group of order 288. The 480 operations of $G_5$ listed in the sets (i) to (xiv) account for 240 of these permutations, and it will be convenient to list typical permutations of each type that arises. The notation of § 5-2 is used for the nodes in $\Pi$, and the types of permutation are referred to as $(a), (b), ..., (i)$; with an indication, for each type of permutation, of those of the sets (i), (ii), ..., of operations of $G_5$ from which the permutation arises. The details of the calculation (which are elementary) are left to the reader. The 240 permutations of the nodes in $\Pi$ are

(a) the identical permutation [(i), (ii)];

(b) 12 permutations such as $(A_2 A_4)(B_2 B_3)(C_2 C_4)(D_2 D_4)$, which keep fixed all the vertices of one tetrahedron and interchange in pairs the vertices of the other tetrahedra collinear with one vertex of the former (in this case the vertex $A_1$) [(iii), (iv)];

(c) 9 permutations such as $(A_3 B_2)(C_2 D_3)(A_3 B_3)(C_3 D_3)$ which leave invariant the vertices of one tetrahedron and interchange the vertices of a pair of opposite edges of each of the other two, the two edges of one of these tetrahedra meeting the two edges of the other [(v)];

(d) 32 permutations such as $(A_1 A_2 A_3)(B_1 B_2 B_3)(C_1 C_2 C_3)(D_1 D_2 D_3)$ permuting the tetrahedra cyclically, one cycle of three nodes (here $A_1 A_2 A_3$) lying on a $\kappa$-line and the others in the three cross-planes passing through the $\kappa$-line [(vi), (vii)];

(e) 36 permutations such as $(A_1 B_1 A_4)(B_1 D_1 C_3)(A_2 B_2)(C_2 D_2)$ which interchange in pairs the vertices on a pair of opposite edges of one of the tetrahedra and permute cyclically the other nodes in $\Pi$ which lie in two cross-planes, one through each of the edges, in such a way that $A_1 B_3, B_2 B_1 ...$ are $\kappa$-lines [(viii)];

(f) 6 permutations such as $(A_1 C_1)(B_1 D_1)(A_2 B_2)(C_2 D_2)(A_3 D_3)(B_3 C_3)$ which interchange the nodes on three mutually skew pairs of opposite edges of the tetrahedra [(ix)];

(g) 16 permutations such as $(A_1 B_2 B_3)(B_1 D_2 C_3)(C_1 C_2 A_4)(D_1 A_2 D_3)$ permuting the tetrahedra cyclically, the four cycles of three nodes belonging to four mutually skew $\kappa$-lines, whose polars lie in a Jacobian plane through $O$ [(x), (xi)];

(h) 32 permutations such as $(B_1 D_1 C_1)(A_2 C_2 D_3)(A_3 C_3 D_3)$ keeping fixed the nodes on a $\kappa$-line (here $A_1 B_2 B_3$) and permuting cyclically the three other vertices of each tetrahedron [(xii), (xiv)];

(i) 96 permutations such as $(A_1 B_2)(B_1 D_2 D_1 A_2 C_1 C_2)(A_3 D_3 C_4)$ which keep fixed a vertex (here $B_3$) of one tetrahedron, permute cyclically the other vertices of this
tetrahedron, interchange two vertices of the other tetrahedra lying on a \( \kappa \)-line through the fixed vertex, and permute the other nodes in a cycle of six in such a way that consecutive edges of the hexagon \( B_1 D_2 D_1 A_2 C_1 C_2 \) pass through

\[ C_3, A_3, D_3, C_3, A_3, D_3 \]

where the cyclic order is the same as that in which \( A_3, D_3, C_3 \) are permuted [(xiii), (xiv)].

It will be proved that the remaining operations of \( G_6 \) consist of:

(xv) 96 operations of the form \( T_1 T_2 \), where \( T_1 \) is a product of projections from three non-collinear nodes in a Jacobian plane through \( O \) which leaves \( O \) invariant, and \( T_2 \) is a product of projections from two vertices of one of the three tetrahedra is \( \Pi \), the product of any such operation with \( p(O) \) being of the same type.

Further, it will be shown that the permutations effected by these operations on the nodes in \( \Pi \) consist of:

\( (j) \) 48 permutations of the form \((A_1 B_2 D_3 B_1 D_2 A_3)(C_1 C_2 C_3 D_1 A_2 B_3)\) permuting the vertices of the three tetrahedra in two cycles of six, the square of the permutation being a permutation of the set \( (g) \).

In the first place, if \( T_2 \) is the product of two projections from nodes on an edge of one of the tetrahedra, then \( T_2 p(O) \) is the product of projections from the two nodes of the opposite edge. Thus if \( T_1 T_2 \) is an operation of the type considered, so also is \( T_1 T_2 p(O) \). Moreover, the permutations effected by \( T_1 \) and \( T_2 \) on the nodes in \( \Pi \) have the forms \( (g) \) and \( (c) \) respectively, and the product of the typical permutations given for these forms is the permutation given for the form \( (j) \). An examination of the nature of the permutations \( (g) \) and \( (c) \) readily shows that the two permutations chosen are typical of any such pair. Moreover, it is easily seen that precisely 48 permutations of type \( (j) \) exist, three having any given permutation of the form \( (g) \) as square; and this indicates the existence of 96 operations of the set (xv), two giving rise to each of the 48 permutations of the nodes in \( \Pi \), these being of the forms \( T_1 T_2, T_2 T_1 p(O) \).

Further study of the operations of this type, which now follows, makes it clear that the operations are completely symmetrical with respect to the three tetrahedra, and that each product \( T_1 T_2 \) can be expressed in three equivalent forms, in which \( T_2 \) is associated with each of the three tetrahedra in turn. A direct verification of this, by means of elementary transformations, is left to the reader. The relation of these operations to the whole set of nodes will now be considered in detail. Take, as a typical operation of this type, the product

\[ L = p(14) p(12, 56, 34) p(16, 35, 24) \cdot p(16) p(16, 45, 23), \]

where the first three factors form an operation \( T_1 \) of type VIII keeping the node \( (45) \) and a Jacobian plane through it fixed, and the last two factors form an operation \( T_2 \), the product of projections from two nodes in the polar prime of \( (45) \). By means of
elementary transformations the product can be transformed into a rather more convenient form. Thus

\[ p(14) p(12, 56, 34) p(16, 35, 24) p(16) p(16, 45, 23) \]
\[ = p(46) p(41) p(15, 34, 26) p(16, 35, 24) p(16, 45, 23) \]
\[ = p(46) p(41) p(15, 34, 26) p(34) p(16, 35, 24) \]
\[ = p(46) p(41) p(43) p(15, 34, 26) p(16, 35, 24), \]

and so

\[ L = p(46) p(41) p(43) p(15, 34, 26) p(16, 35, 24), \]

which will be regarded henceforth as the standard form. Now

\[ L^2 = p(46) p(41) p(43) p(15, 34, 26) p(16, 35, 24) \]
\[ \times p(46) p(41) p(43) p(15, 34, 26) p(16, 35, 24) \]
\[ = [p(46) p(41) p(43)]^2 p(12, 35, 46) p(13, 45, 26) p(15, 34, 26) p(16, 35, 24) \]
\[ = p(16) p(36) p(41) p(43) p(12, 35, 46) p(16, 35, 24) p(14, 56, 23) p(14, 36, 25) \]
\[ = p(36) p(31) p(31) p(41) p(26) p(12, 35, 46) p(14, 56, 23) p(14, 36, 25) \]
\[ = p(36) p(41) p(26) p(14, 56, 23) p(12, 35, 46) p(14, 36, 25) \]
\[ = p(26) p(23) p(41) p(14, 56, 23) p(12, 35, 46) p(14, 36, 25) \]
\[ = p(26) p(56) p(14, 23, 56) p(12, 35, 46) p(14, 36, 25) \]
\[ = p(25) p(25) p(14, 23, 56) p(14, 36, 25) p(12, 35, 46) \]
\[ = p(25) p(14, 23, 56) p(12, 35, 46), \]

which is easily recognized as an operation of type VIII leaving (45) invariant, and

\[ L^3 = p(46) p(41) p(43) p(15, 34, 26) p(16, 35, 24) p(25) p(14, 23, 56) p(12, 35, 46) \]
\[ = p(46) p(41) p(43) p(25) p(12, 34, 56) p(16, 23, 45) p(14, 23, 56) p(12, 35, 46) \]
\[ = p(16) p(36) p(25) p(46) p(16, 23, 45) p(14, 23, 56) p(26) p(12, 35, 46) \]
\[ = p(16) p(36) p(25) p(26) p(12, 35, 46) p(12, 35, 46) \]
\[ = p(16) p(36) p(25) p(26) p(12, 35, 46) \]
\[ = p(16) p(36) p(25) p(26) p(56) p(12, 36, 45) \]
\[ = p(16) p(36) p(25) p(26) p(12, 36, 45), \]

which is an operation of type X whose square is the projection \( p(45) \). Thus \( L \) is an operation of period 12, and \( L^6 = p(45) \).

The operation \( L \) keeps the node (45) fixed; it permutes the nodes in the polar prime of (45) in two cycles of six, and the remaining 32 nodes in two cycles of four and two cycles of twelve, the actual cycles being

\[
[(12), (16), (13, 26, 45), (36), (16, 45, 23), (13)], \\
[(23), (26), (12, 45, 36), (16, 23, 45), (13, 45, 26), (12, 36, 45)], \\
[(24), (12, 34, 56), (25), (12, 35, 46)], \\
[(14, 23, 56), (15, 34, 26), (15, 23, 46), (14, 35, 26)], \\
[(14), (15, 26, 34), (13, 56, 24), (12, 56, 34), (16, 25, 34), (14, 56, 23), (15), \\
(14, 26, 35), (13, 46, 25), (12, 46, 35), (16, 24, 35), (15, 46, 23)], \\
[(34), (16, 24, 25), (15, 24, 36), (15, 36, 24), (13, 25, 46), (46), (35), (16, 35, 24), \\
(14, 25, 36), (14, 36, 25), (13, 24, 56), (56)].
\]

† Since the nodes (14), (23), (56), (14, 23, 56), (14, 56, 23) form a Jordan pentahedron.
Since $L$ keeps only one node fixed the operations of this type arising from different nodes are distinct. Since the squares of these operations are of type VIII, the aggregate of operations of this type forms at least two conjugate sets in $G$. It will be proved that there are exactly two such sets. For the eight nodes belonging to the two cycles of four lie in a Jacobian plane through (45). The four nodes in any one of these cycles lie alternately on two $\kappa$-lines through (45), and the joins of consecutive nodes pass, in order, through the four nodes of the other cycle. Thus either cycle determines the other. There are six pairs of cycles of this kind associated with each Jacobian plane through (45), and it is easy to see that there is a collineation in $G$ which transforms any one of them into any other. Thus, of the 96 operations such as $L$ which keep (45) fixed, there are two associated with each of the 48 pairs of cycles, and if one of these is $L$ the other is clearly $L^5$. Then every operation of the same type as $L$ is conjugate to one of the two operations $L, L^5$. But these are not conjugate, since their squares are non-conjugate operations of type VIII. Thus $G$ contains

XV. 4320 operations of period 12, products of five projections, forming two conjugate sets of 2160 operations each, whose squares are of type VIII and whose cubes are of type X.

Since the types I to XV include 25920 distinct operations, all the operations of $G$ are accounted for, and it follows that every operation of $G$ is expressible as the product of five or fewer projections. The distribution of the operations of $G$ in conjugate sets agrees with that obtained, by very different methods, by Frame (1936, p. 483).

7. Some subgroups of $G$

As an illustration of the usefulness of the classification of operations of $G$ undertaken above it will be shown how the nature of the operations of five important maximal subgroups of $G$ can be determined. These five subgroups are:

(i) the subgroup $G_1$ of order 960 leaving invariant a Jordan pentahedron,
(ii) the subgroup $G_2$ of order 720 leaving invariant the nodes of a 15-set,
(iii) the subgroup $G_3$ of order 648 leaving invariant a Steiner prime,
(iv) the subgroup $G_4$ of order 648 leaving invariant a Jacobian plane,
(v) the subgroup $G_5$ of order 576 leaving invariant a node.

Of these, $G_3$ and $G_5$ have been described in §§ 5–3 and 6 respectively. The procedure for determining the operations of the other subgroups will be given in outline, leaving the details to the reader and referring him to the table at the end of the section for the results.

For $G_1$ one observes first that there are 192 operations which transform a Jordan pentahedron into itself and keep a node of this fixed, these forming a subgroup of index 3 in the $G_5$ which keeps the node fixed. These operations are easily identified in the list of operations of $G_5$ already given. By examining the permutations effected on the nodes of the Jordan pentahedron by these operations it is easy to see, first, that from the five vertices of the pentahedron 576 distinct operations are obtained in this way, and secondly, that these operations all effect even permutations of the
five nodes of the pentahedron. Hence, as \( 576 = \frac{3}{2} \cdot 960 \), every operation of \( G_1 \) permutes the vertices of the pentahedron evenly. Any of the remaining 384 operations of \( G_1 \) thus effects an even permutation of the nodes of the pentahedron which keeps no vertex fixed. Such a permutation must therefore consist of a single cycle of five vertices, and the corresponding operation of \( G_1 \) must be of type IX, the only type whose period is divisible by 5.

For \( G_3 \) it is sufficient to observe that, from a consideration of the 15-set consisting of the nodes \((ij)\), the group is simply isomorphic with the symmetric group of degree 6, and is generated by the projections \( p(ij) \) which correspond to transpositions. The only operation of \( G_3 \) whose nature is not at once clear is the cyclic operation of period 6, and it may be seen, either by reducing the product of projections corresponding to such an operation, or (more simply) by considering the natures of the square and cube of the operation, that it is actually of type XII.

For \( G_4 \) there have been given in § 4·3 54 operations generated by projections from the nodes lying in a Jacobian plane, and from the list of operations of \( G_3 \) given in § 5·3 one can easily select a further 108 operations, associated with each Steiner prime through the plane, which leave the plane invariant. These account in all for 486 operations of \( G_4 \), and the remaining operations are easily seen to consist of 108 operations of type XV and 54 of type X which are the cubes of these.

In table 1 are enumerated the operations of each of the fifteen types in \( G \) and its subgroups \( G_1, G_2, G_3, G_4, G_5 \), and also, for each type of operation, its period \( p \) and the least number \( n \) of projections whose product is of the given type. Each type of operation corresponds to a single conjugate set in \( G \) with the exception of those marked by an asterisk, which each yield two conjugate sets.

<table>
<thead>
<tr>
<th>type</th>
<th>( p )</th>
<th>( n )</th>
<th>( G )</th>
<th>( G_1 )</th>
<th>( G_2 )</th>
<th>( G_3 )</th>
<th>( G_4 )</th>
<th>( G_5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>1</td>
<td></td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>II</td>
<td>2</td>
<td>1</td>
<td>45</td>
<td>5</td>
<td>15</td>
<td>18</td>
<td>9</td>
<td>13</td>
</tr>
<tr>
<td>III</td>
<td>2</td>
<td>2</td>
<td>270</td>
<td>70</td>
<td>60</td>
<td>27</td>
<td></td>
<td>30</td>
</tr>
<tr>
<td>IV</td>
<td>3</td>
<td>2</td>
<td>480</td>
<td></td>
<td>40</td>
<td>84</td>
<td>48</td>
<td>32</td>
</tr>
<tr>
<td>V</td>
<td>6</td>
<td>3</td>
<td>1440</td>
<td></td>
<td>120</td>
<td>36</td>
<td>72</td>
<td>32</td>
</tr>
<tr>
<td>VI</td>
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<td>3</td>
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<td>120</td>
<td>180</td>
<td></td>
<td>102</td>
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</tr>
<tr>
<td>VII*</td>
<td>3</td>
<td>4</td>
<td>80</td>
<td></td>
<td></td>
<td>8</td>
<td>26</td>
<td>16</td>
</tr>
<tr>
<td>VIII*</td>
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<td>720</td>
<td></td>
<td></td>
<td>72</td>
<td>90</td>
<td>16</td>
</tr>
<tr>
<td>IX</td>
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<td>4</td>
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<td>384</td>
<td>144</td>
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<tr>
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<tr>
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<td>40</td>
<td>6</td>
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<td></td>
<td></td>
<td>144</td>
<td>144</td>
<td></td>
</tr>
<tr>
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<td>5</td>
<td>1440</td>
<td>160</td>
<td></td>
<td>36</td>
<td>72</td>
<td>128</td>
</tr>
<tr>
<td>XV*</td>
<td>12</td>
<td>5</td>
<td>4320</td>
<td></td>
<td></td>
<td></td>
<td>108</td>
<td>96</td>
</tr>
</tbody>
</table>

\[
\begin{array}{llllll}
25920 & 960 & 720 & 648 & 648 & 576
\end{array}
\]
8. SOME SETS OF GENERATORS OF $G$

I shall conclude by showing how the expressions of the operations of $G$ in terms of projections can be usefully employed in investigating various systems of generators of the group. It does not seem unreasonable to suppose that a systematic investigation of such sets could be made along these lines; but no such investigation (which would of necessity be elaborate) has yet been made, and I content myself with a few illustrative examples, some of them new and others confirming results already obtained.

The group $G$ is generated by the 45 projections (or, indeed, as it is simple, by the operations of any one of its conjugate sets). But these are connected by many relations, and it is natural to inquire as to the minimum number of projections in terms of which all the others can be expressed (and hence any operation of $G$ can be expressed). This number must be five at least, since any four nodes determine a prime, and it has already been seen that the projections from the nodes which lie in any prime generate a proper subgroup of $G$. In fact, five projections suffice. This is shown by proving that any projection can be expressed in terms of the projections from the five nodes

$$(12), (16), (25), (46), (15, 26, 34),$$

(which are clearly typical of numerous other similar sets). The proof rests on the identities

$$p(ij)p(jk)p(ij) = p(ij),$$
$$p(ij)p(ik,jl,mn)p(ij) = p(jk, il, mn),$$
$$p(ij)p(kl)p(mn) = p(ij, kl, mn)p(ij, mn, kl),$$

which are consequences of the fundamental relations (1) between the projections given in § 2 and the relations of incidence between the nodes. From the first of these it is easy to see that every $p(ij)$ for which $i, j \neq 3$ is expressible in terms of $p(12), p(16), p(25), p(46)$—in fact these nodes $(ij)$ are just the nodes of an $n$-prime. From the second identity it easily follows that every $p(ij, kl, mn)$ is expressible in terms of these ten projections and $p(15, 26, 34)$. Whence, from the third identity, $p(i3)$ can be found, proving the theorem.

Coxeter has shown (Coxeter 1940, p. 474) that the relations

$$U^3 = V^2 = W^2 = X^2 = Y^2 = (UV)^3 = (VW)^3 = (WX)^3 = (XY)^3$$
$$= (UW)^2 = (UX)^2 = (UY)^2 = (VX)^2 = (WX)^2 = (XY)^2 = 1$$

constitute an abstract definition for $G$ in terms of five suitably chosen operations $U, V, W, X, Y$. These generating relations are satisfied by taking

$$U = p(23)p(12)p(56)p(45), \quad V = p(25)p(36), \quad W = p(13)p(45),$$
$$X = p(23)p(56), \quad Y = p(16, 25, 34)p(15, 36, 24).$$

Here we recognize $U$ as an operation of type XI and period 3, and $V, W, X, Y$ as operations of type III and period 2. That the other generating relations are satisfied
is most simply shown by reducing the products of pairs of these operations to one of
the standard types by means of elementary transformations. Two examples of the
process will suffice. Thus

\[ XY = p(23)p(56)p(16, 25, 34)p(15, 36, 24) = p(15, 36, 24)p(23)p(56)p(15, 36, 24) \]
\[ = p(15, 26, 34)p(16, 35, 24), \]
an operation of type III and period 2, and

\[ UV = p(23)p(12)p(56)p(45)p(25)p(36) = p(13)p(23)p(56)p(25)p(24)p(36) \]
\[ = p(13)p(23)p(25)p(26)p(36)p(24) = p(13)p(35)p(26)p(24), \]
an operation of type XI and period 3. The remaining relations may be verified in a
similar way.

That these five operations generate \( G \) follows from the stronger theorem that \( G \)
is generated by the two operations \( H = UVW \) and \( K = XY \). That \( G \) possesses a
pair of generating operations seems to have been first pointed out by Brahma
(1930, p. 533), and the identification of Brahma's operations with \( H \) and \( K \) is stated,
without proof, by Coxeter (1940, p. 475). A simple proof of the fact that \( H \) and \( K \)
generate \( G \) will now be given. Since

\[ UVW = p(13)p(35)p(26)p(24).p(13)p(45) = p(15)p(26)p(24)p(45) \]
\[ = p(42)p(46)p(15)p(45) = p(42)p(46)p(45)p(41), \]
an operation of period 5 and type IX, it follows that

\[ H = p(42)p(46)p(45)p(41), \quad K = p(15, 26, 34)p(16, 35, 24), \]
and

\[ HK = p(42)p(46)p(45)p(41).p(15, 26, 34)p(16, 35, 24) \]
\[ = p(46)p(41)p(26)p(15)p(15, 26, 34)p(16, 35, 24) \]
\[ = p(46)p(41)p(43)p(15, 34, 26)p(16, 35, 24), \]

which is the operation \( L \) of type XV and period 12 considered in §6. Thus the group
generated by \( H \) and \( K \) contains \( L \), and hence contains \( L^6 = p(45) \). It was shown
in §2 that if an operation \( T \) of \( G \) transforms a node \( A \) into a node \( A' \), then

\[ Tp(A)T^{-1} = p(A'). \]

Hence, since \( H \) permutes the nodes in such a way that the digits in their symbols
undergo the permutation \( (15624) \), the operations \( H^rL^6H^{-r} \), for \( r = 1, 2, 3, 4 \)
are respectively \( p(16), p(25), p(46), p(12) \). And hence \( p(14) = p(16)p(46)p(16) \) belongs
to the group generated by \( H, K \), and so does \( Lp(14)L^{-1} \), which is \( p(15, 26, 34) \), since \( L \)
transforms the node \( (14) \) into \( (15, 26, 34) \) (from the expression of \( L \) as a permutation
of the nodes given in §8). Thus the group generated by \( H \) and \( K \) contains the
projections from the five nodes \( (12), (16), (25), (46), (15, 26, 34) \), and it was seen at the
beginning of the present section that these projections generate the entire group.

\[ \dagger \text{ Since } (15), (26), (34), (15, 26, 34), (15, 34, 26) \text{ form a Jordan pentahedron.} \]
It is possible to show that \( G \) can also be generated by the two operations \( L \) and \( p(A) \), where \( A \) is one of the nodes belonging to a cycle of twelve permuted by \( L \).

It is easy to verify that, in the first of the two cycles listed in § 6, the join of any two consecutive nodes in the cycle is a \( \kappa \)-line, whose third node is one of those permuted by \( L \) in a cycle of six, and that in the operation \( L^7 \) the second cycle has this property. Since \( (L^7)^7 = L^49 = L \) it follows that there is nothing lost in generality by supposing that \( A \) belongs to the first cycle. And since the nodes in any one cycle are permuted cyclically by \( L \), the projections from these nodes are all of the form \( L^TTL^{-r} \), where \( T \) is any one of them. It is thus sufficient to prove that \( L \) and \( p(14) \) generate the group. From what has been said, the group generated by \( L \) and \( p(14) \) contains the projections from all 12 nodes of the cycle to which \( (14) \) belongs. And from the projections from two consecutive nodes in the cycle can be deduced the projection from the third node on the \( \kappa \)-line joining them, which is a node belonging to the second of the two cycles of six. And the projection \( L^5 = p(45) \) also belongs to the group. Thus, in particular, the group generated by \( L \) and \( p(14) \) contains the projections from the nodes \((23), (26)\) and hence the projections from \( (36) \) and the remaining nodes of the first cycle of six to which \( (36) \) belongs. Thus the group contains the five projections \( p(12), p(14), p(23), p(26), p(15, 26, 34) \), and these are easily seen to form a set of generators of \( G \) similar to that given at the beginning of the section.

It will be noticed that

\[
Lp(14) = p(46) p(41) p(43) p(15, 34, 26) p(16, 35, 24) p(14)
\]

\[
= p(46) p(13) p(12, 46, 35) p(12, 46, 35),
\]

and it is easy to see that the four nodes concerned in these four projections determine an \( \eta \)-prime, so that \( Lp(14) \) is an operation of period 5 and type IX. The operation \( L^{-1}p(14) \), whose inverse \( p(14) L \) is conjugate to \( Lp(14) \), is of the same type. Also,

\[
\]

\[
= p(56) p(45) p(13) p(12, 46, 35)
\]

\[
= p(56) p(26) p(12, 46, 35) p(13, 45, 26)
\]

\[
= p(56) p(26) p(12, 46, 35) p(13, 45, 26).
\]

The product of the first three factors is easily seen to be an operation of type VIII, associated with a Jacobian plane containing the nodes \((56), (26), (12, 46, 35)\), which keeps fixed the node \((16, 45, 23)\). And since this node is joined by a \( \kappa \)-line to \((13, 45, 26)\) it follows that \( L^7p(14) \) is an operation of type XIII and period 9. So, also, \( L^9p(14) \), which is conjugate to \( p(14) L^5 = [L^7p(14)]^{-1} \), is of period 9.

Hence \( Lp(14) \) and \( L^7p(14) \) have periods 5 and 9. It is easily seen that the same is true if \( p(14) \) be replaced by the projection from any other node in the cycle of twelve

\[\dagger\] Since \((13), (26), (45), (13, 26, 45), (13, 45, 26)\) form a Jordan pentahedron.
to which (14) belongs, and that for projections from nodes of the other cycle of twelve the corresponding periods are 9 and 5.

Since $L$ and $p(14)$ generate $G$ so also do either of the pairs of operations $Lp(14)$, $p(14)$ or $L^7p(14)$, $p(14)$, that is, $G$ admits a pair of generators of periods 5, 2 or 9, 2 whose product is of period 12. Frame has given, as a pair of generators of $G$, two operations of periods 9 and 2 whose product has period 5 (Frame 1936, p. 482). It is easy to find such a pair. In fact, $G$ is generated by the two operations $L^7p(14)$ and $L^6 = p(45)$, whose product $Lp(14)$ has period 5. To see this, write

$$P = L^7p(14), \quad Q = L^6, \quad R = QP = Lp(14).$$

Then $R$ belongs to the group generated by $P$ and $Q$. Moreover,

$$P^{-1}R = p(14) L^{-7}. Lp(14) = p(14) L^6p(14) = p(14)p(45)p(14) = p(15),$$

and hence

$$P^{-1}RQ = p(15) L^6 = p(15)p(45) = p(45)p(14) = L^6p(14) = Qp(14).$$

Thus $QP^{-1}RQ = p(14)$ and so $p(14)$ and $Rp(14) = L$ belong to the group generated by $P$ and $Q$. Thus, since $L$ and $p(14)$ generate $G$, so also do $P$ and $Q$.

A closely related pair of generators has been given by the author (Todd 1936, p. 173), namely, a pair of operations $S$ and $T$ each of period 5, such that $ST$ is of period 9 and $ST^{-1}$ is of period 3. Such a pair of operations is given in fact by

$$S = QP, \quad T = PQ,$$

where $P$ and $Q$ are the operations just described. For these are of period 5, and $ST = QP^2Q$ is conjugate to $P^2$ (and so of period 9), while

$$ST^{-1} = QPQ^{-1}P^{-1} = Lp(14)p(45)p(14) L^{-7} = L[p(14)p(45)]^2 L^{-1},$$

which is of period 3. And, since $Q^2 = 1$, $TS = P^2$ and $(TS)^5 = P$. Thus

$$P = (TS)^5, \quad Q = (TS)^{-5} T,$$

are expressible in terms of $S$ and $T$. Whence, as $P$ and $Q$ generate $G$, so also do $S$ and $T$.

It seems rather striking that the pairs of generators found, in very different ways, by various writers, should prove to be related to each other in so simple a manner.

I should like to express my thanks to Professor Baker, who has very kindly placed the proof sheets of his tract at my disposal, and with whom I have had many stimulating discussions on the subject of this paper.
A technique is described for the investigation of gaseous spectra at high radio-frequencies. The sensitivity of the method is such that an absorption of $2 \times 10^{-6}$ per cm. of path can be detected at wave-lengths around 1 cm.

The inversion spectrum of ammonia in the ground vibrational state has been examined at wave-lengths between 1.1 and 1.6 cm. and found to contain an extensive fine structure, arising from centrifugal distortion of the molecule. Twenty-nine lines have been identified, each corresponding to a different rotational quantum state. The wave numbers of these lines can be accurately represented by the formula

$$
\tilde{v} = 0.7935 + [-0.0050 J^2 + J + 0.0070 K^2] \\
+ 0.63[-0.0050 J^2 + J + 0.0070 K^2]^2 \text{ cm.}^{-1} \text{ (vacuo)}
$$

where $J, K$ are the rotational quantum numbers of the symmetrical top molecule.

Detailed measurements of 18 lines at a pressure of 0.5 mm. Hg show that their half widths, due to collision broadening, lie between 2 and $5 \times 10^{-4}$ cm.$^{-1}$, while their intensities agree with the theoretical values, within the experimental error of $\pm 5\%$. The integrated intensity at a pressure of 4.5 mm. Hg, over the interval 0.67 to 0.87 cm.$^{-1}$, agrees with the calculated intensity to better than 1%.

1. INTRODUCTION

The well-known doubling of most of the lines of the infra-red spectrum of gaseous ammonia is associated with the existence of two equilibrium configurations of the molecule, with the nitrogen atom on either side of the plane containing the three hydrogen atoms. The appropriate potential energy curve of the molecule has two identical minima, separated by a fairly low hump; the associated ‘resonance’

References


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