

Research



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One contribution to a Special feature 'Climate dynamics: multiple scales, memory effects and control perspectives'.

Tropical atmospheric circulations with humidity effects

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The main objective of this article is to study the effect of the moisture on the planetary scale atmospheric circulation over the tropics. The modelling we adopt is the Boussinesq equations coupled with a diffusive equation of humidity, and the humidity-dependent heat source is modelled by a linear approximation of the humidity. The rigorous mathematical analysis is carried out using the dynamic transition theory. In particular, we obtain mixed transitions, also known as random transitions, as described in Ma & Wang (2010 *Discrete Contin. Dyn. Syst.* **26**, 1399–1417. (doi:10.3934/dcds.2010.26.1399); 2011 *Adv. Atmos. Sci.* **28**, 612–622. (doi:10.1007/s00376-010-9089-0)). The analysis also indicates the need to include turbulent friction terms in the model to obtain correct convection scales for the large-scale tropical atmospheric circulations, leading in particular to the right critical temperature gradient and the length scale for the Walker circulation. In short, the analysis shows that the effect of moisture lowers the magnitude of the critical thermal Rayleigh number and does not change the essential characteristics of dynamical behaviour of the system.

1. Introduction

This article is part of a research programme to study low-frequency variability of the atmospheric and oceanic flows. As we know, typical sources of

climate low-frequency variability include the wind-driven (horizontal) and thermohaline (vertical) circulations (THC) of the ocean, and the El Niño Southern Oscillation (ENSO). Their variability, independently and interactively, may play a significant role in climate change, past and future. The primary goal of our study is to document, through careful theoretical and numerical studies, the presence of climate low-frequency variability, to verify the robustness of this variability's characteristics to changes in model parameters and to help explain its physical mechanisms. The thorough understanding of the variability is a challenging problem with important practical implications for geophysical efforts to quantify predictability, analyse error growth in dynamical models and develop efficient forecast methods.

ENSO is one of the strongest interannual climate variabilities associated with strong atmosphere–ocean coupling, with significant impacts on global climate. ENSO consists of warm events (El Niño phase) and cold events (La Niña phase) as observed by the equatorial eastern Pacific sea-surface temperature (SST) anomalies, which are associated with persistent weakening or strengthening in the trade winds; see among others [1–16]. An interesting current debate is whether ENSO is best modelled as a stochastic or chaotic system—linear and noise-forced, or nonlinear oscillatory and unstable system [14]. It is obvious that a careful fundamental level examination of the problem is crucial. For this purpose, Ma & Wang [17] initiated a study of ENSO from the dynamical transition point of view and derived in particular a new oscillation mechanism of ENSO. Namely, ENSO is a self-organizing and self-excitation system, with two highly coupled oscillation processes—the oscillation between metastable El Niño and La Niña and normal states, and the spatio-temporal oscillation of the SST.

The main objective of this article is to address the moisture effect on the low-frequency variability associated with ENSO. First, as our main purpose is to capture the patterns and general features of the large-scale atmospheric circulation over the tropics, it is appropriate to use the Boussinesq equations coupled with a diffusive equation of humidity. In addition, the humidity effect is also taken into consideration by treating the heating source as a linear approximation of the humidity function.

Second, although the introduction of the humidity effect leads to substantial difficulty from the mathematical point of view, we have shown in theorem 4.1 that the humidity does not affect the type of dynamic transition the system undergoes. Namely, we show that under the idealized boundary conditions, only continuous transition (Type I) occurs. However, the critical thermal Rayleigh number is slightly smaller than that in the case without moisture factor. To see this effect of humidity, we refer to formula (3.31), which reads

$$R_c = - \left[\frac{1}{Le} + \left(1 + \frac{Le}{24} \alpha_{k_c}^2 \right) \frac{\alpha_1 \alpha_T h^2}{Le \alpha_q \alpha_{k_c}^2 \kappa_T} \right] \tilde{R} + \alpha_{k_c}^4 \frac{r_0^2}{k^2} \left(\alpha_{k_c}^2 + \frac{1}{r_0^2} \right) + \frac{\alpha_{k_c}^2}{r_0^2}.$$

As in the expression of the thermal critical Rayleigh number R_c , the coefficient of the humidity Rayleigh number \tilde{R} is negative; this indicates that the presence of humidity lowers the critical temperature difference for the onset of the dynamic transition.

Third, we remark that the perturbation analysis in [17] can be applied to the case here to carry out the analysis, and we can show that under the natural boundary condition, the underlying system with humidity effect will undergo a mixed-type transition. In addition, as we argued in [17], it is necessary to include turbulent friction terms in the model to obtain correct convection scales for the large-scale tropical atmospheric circulations, leading in particular to the right critical temperature gradient and the length scale for the Walker circulation.

Finally, based on these theoretical results, it is easy then to conclude the same mechanism for ENSO as proposed in Ma & Wang [17,18]. In particular, the random transition behaviour of the system explains general features of the observed abrupt changes between strong El Niño and strong La Niña states. Also, with the deterministic model considered in this article, the randomness is closely related to the uncertainty/fluctuations of the initial data between the narrow basins of attractions of the corresponding metastable events, and the deterministic feature is represented by a deterministic coupled atmospheric and oceanic model predicting the basins

of attraction and the SST. In addition, from the predictability and prediction point of view, it is crucial to capture more detailed information on the delay feedback mechanism of the SST. For this purpose, the study of an explicit multi-scale coupling mechanism to the ocean is inevitable. In fact, the new mechanism strongly suggests the need and importance of the coupled ocean–atmosphere models for ENSO predication in [1,3–11].

This article is organized as follows. Section 2 gives the objective Boussinesq model with humidity. The eigenvalue problem is analysed in §3. The transition theorem is stated and proved in §4. In §5, the turbulence friction factors are considered and the corresponding critical temperature difference and the wavenumbers of Walker circulation are checked.

2. Model for atmospheric motion with humidity

(a) Atmospheric circulation model

The hydrodynamical equations governing the atmospheric circulation is the Navier–Stokes equations with the Coriolis force generated by the Earth’s rotation, coupled with the first law of thermodynamics.

Let (φ, θ, r) be the spheric coordinates, where φ represents the longitude, θ the latitude and r the radial coordinate. The unknown functions include the velocity field $u = (u_\varphi, u_\theta, u_r)$, the temperature function T , the humidity function q , the pressure p and the density function ρ . Then the equations governing the motion and states of the atmosphere consist of the momentum equation, the continuity equation, the first law of thermodynamics, the diffusion equation for humidity and the equation of state (for ideal gas), which read

$$\rho \left[\frac{\partial u}{\partial t} + \nabla_u u + 2\Omega \times u \right] + \nabla p + k\rho g = \mu \Delta u, \quad (2.1)$$

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho u) = 0, \quad (2.2)$$

$$\rho c_v \left[\frac{\partial T}{\partial t} + u \cdot \nabla T \right] + p \operatorname{div} u = \tilde{Q} + \tilde{\kappa}_T \Delta T, \quad (2.3)$$

$$\rho \left[\frac{\partial q}{\partial t} + u \cdot \nabla q \right] = \tilde{S} + \tilde{\kappa}_q \Delta q \quad (2.4)$$

and

$$p = R\rho T. \quad (2.5)$$

Here, $0 \leq \varphi \leq 2\pi$, $-\pi/2 \leq \theta \leq \pi/2$, $a < r < a + h$, a is the radius of the Earth, h is the height of the troposphere, Ω is the Earth’s rotating angular velocity, g is the gravitational constant, μ , $\tilde{\kappa}_T$, $\tilde{\kappa}_q$, c_v , R are constants, \tilde{Q} and \tilde{S} are heat and humidity sources, and $k = (0, 0, 1)$. The differential operators used are as follows:

- (1) The gradient and divergence operators are given by

$$\nabla = \left(\frac{1}{r \cos \theta} \frac{\partial}{\partial \varphi}, \frac{1}{r} \frac{\partial}{\partial \theta}, \frac{\partial}{\partial r} \right)$$

and

$$\operatorname{div} u = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 u_r) + \frac{1}{r \cos \theta} \frac{\partial (u_\theta \cos \theta)}{\partial \theta} + \frac{1}{r \cos \theta} \frac{\partial u_\varphi}{\partial \varphi}$$

- (2) In the spherical geometry, although the Laplacian for a scalar is different from the Laplacian for a vectorial function, we use the same notation Δ for both of them

$$\begin{aligned} \Delta u = & \left(\Delta u_\varphi + \frac{2}{r^2 \cos \theta} \frac{\partial u_r}{\partial \varphi} + \frac{2 \sin \theta}{r^2 \cos^2 \theta} \frac{\partial u_\theta}{\partial \varphi} - \frac{u_\varphi}{r^2 \cos^2 \theta}, \right. \\ & \left. \Delta u_\theta + \frac{2}{r^2} \frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r^2 \cos^2 \theta} - \frac{2 \sin \theta}{r^2 \cos^2 \theta} \frac{\partial u_\varphi}{\partial \varphi}, \right. \end{aligned}$$

$$\Delta u_r - \frac{2u_r}{r^2} - \frac{2}{r^2 \cos \theta} \frac{\partial(u_\theta \cos \theta)}{\partial \theta} - \frac{2}{r^2 \cos \theta} \frac{\partial u_\varphi}{\partial \varphi}$$

$$\text{and} \quad \Delta f = \frac{1}{r^2 \cos \theta} \frac{\partial f}{\partial \theta} \left(\cos \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \cos^2 \theta} \frac{\partial^2 f}{\partial \varphi^2} + \frac{1}{r^2} \frac{\partial f}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right).$$

(3) The convection terms are given by

$$\begin{aligned} \nabla_u u = & \left(u \cdot \nabla u_\varphi + \frac{u_\varphi u_r}{r} - \frac{u_\varphi u_\theta}{r} \tan \theta, \right. \\ & \left. u \cdot \nabla u_\theta + \frac{u_\theta u_r}{r} + \frac{u_\varphi^2}{r} \tan \theta, u \cdot \nabla u_r - \frac{u_\varphi^2 + u_\theta^2}{r} \right). \end{aligned}$$

(4) The Coriolis term $2\boldsymbol{\Omega} \times u$ is given by

$$2\boldsymbol{\Omega} \times u = 2\Omega(\cos \theta u_r - \sin \theta u_\theta, \sin \theta u_\varphi, -\cos \theta u_\varphi).$$

Here, $\boldsymbol{\Omega}$ is the angular velocity vector of the Earth, and Ω is the magnitude of the angular velocity.

The above system of equations is basically the equations used by L. F. Richardson in his pioneering work [19]. However, they are in general too complicated to conduct theoretical analysis. As practised by earlier researchers such as Charney [20], and from the lessons learned by the failure of Richardson's pioneering work, one tries to be satisfied with simplified models approximating the actual motions to a greater or lesser degree instead of attempting to deal with the atmosphere in all its complexity. By starting with models incorporating only what are thought to be the most important of atmospheric influences, and by gradually bringing in others, one is able to proceed inductively and thereby to avoid the pitfalls inevitably encountered when a great many poorly understood factors are introduced all at once. The simplifications are usually done by taking into consideration some of the main characterizations of the large-scale atmosphere. One such characterization is the small aspect ratio between the vertical and horizontal scales, leading to a hydrostatic equation replacing the vertical momentum equation. The resulting system of equations is called the primitive equations (see among others [21]). Another characterization of large-scale motion is the fast rotation of the Earth, leading to the celebrated quasi-geostrophic equations [22].

(b) Tropical atmospheric circulation model

In this article, our main focus is on formation and transitions of the general circulation patterns. For this purpose, the approximations we adopt involve the following components.

First, we often use the Boussinesq assumption, where the density is treated as a constant except in the buoyancy term and in the equation of state.

Second, because the air is generally not incompressible, we do not use the equation of state for ideal gas; rather, we use the following empirical formula, which can be regarded as the linear approximation of (2.5):

$$\rho = \rho_0[1 - \alpha_T(T - T_0) - \alpha_q(q - q_0)], \quad (2.6)$$

where ρ_0 is the density at $T = T_0$ and $q = q_0$, and α_T and α_q are the coefficients of thermal and humidity expansion.

Third, as the aspect ratio between the vertical scale and the horizontal scale is small, the spheric shell, which the air occupies, is treated as a product space $S_a^2 \times (a, a + h)$ with the product metric

$$ds^2 = a^2 d\theta^2 + a^2 \sin^2 \theta d\phi^2 + dz^2.$$

This approximation is extensively adopted in geophysical fluid dynamics.

Fourth, the hydrodynamic equations governing the atmospheric circulation over the tropical zone are the Navier–Stokes equations coupled with the first law of thermodynamics and the

diffusion equation of the humidity. These equations are restricted on the lower latitude region where the meridional velocity component u_θ is zero.

Let $(\phi, z) \in M = (0, 2\pi) \times (a, a + h)$ be the coordinate, where ϕ is the longitude, a the radius of the Earth and h the height of the troposphere. The unknown functions include the velocity field $u = (u_\phi, u_z)$, the temperature function T , the humidity function q and the pressure p . Then, the equations governing the motion and states of the atmosphere read

$$\left. \begin{aligned} \frac{\partial u_\phi}{\partial t} + (u \cdot \nabla)u_\phi + \frac{u_\phi u_z}{a} &= v \left(\Delta u_\phi + \frac{2}{a^2} \frac{\partial u_z}{\partial \phi} - \frac{u_\phi}{a^2} \right) - 2\Omega u_z - \frac{1}{\rho_0 a} \frac{\partial p}{\partial \phi}, \\ \frac{\partial u_z}{\partial t} + (u \cdot \nabla)u_z - \frac{u_\phi^2}{a} &= v \left(\Delta u_z - \frac{2}{a^2} \frac{\partial u_\phi}{\partial \phi} - \frac{2u_z}{a^2} \right) + 2\Omega u_\phi - \frac{1}{\rho_0} \frac{\partial p}{\partial z} \\ &\quad - g(1 - \alpha_T(T - T_0) - \alpha_q(q - q_0)), \\ \frac{\partial T}{\partial t} + (u \cdot \nabla)T &= \kappa_T \Delta T + Q, \\ \frac{\partial q}{\partial t} + (u \cdot \nabla)q &= \kappa_q \Delta q + S, \\ \text{and } \frac{1}{a} \frac{\partial u_\phi}{\partial \phi} + \frac{\partial u_z}{\partial z} &= 0, \end{aligned} \right\} \quad (2.7)$$

where Ω is the Earth's rotating angular velocity, g the gravitational constant, ρ_0 the density of air at $T = T_0$, $v = \mu/\rho_0$, $\kappa_T = \tilde{\kappa}_T/\rho_0 c_v$, $\kappa_q = \tilde{\kappa}_q/\rho_0$, $Q = \tilde{Q}/\rho_0 c_v$, $S = \tilde{S}/\rho_0$ and α_T and α_q are the coefficients of thermal and humidity expansion. However, the differential operators used in (2.7) are as follows:

$$\begin{aligned} (u \cdot \nabla) &= \frac{u_\phi}{a} \frac{\partial}{\partial \phi} + u_z \frac{\partial}{\partial z}, \\ \Delta f &= \frac{1}{a^2} \frac{\partial^2}{\partial \phi^2} f + \frac{\partial^2}{\partial z^2} f \end{aligned}$$

and

$$\begin{aligned} \Delta u &= \Delta(u_\phi e_\phi + u_z e_z) \\ &= \left(\Delta u_\phi + \frac{2}{a^2} \frac{\partial u_z}{\partial \phi} - \frac{u_\phi}{a^2} \right) e_\theta + \left(\Delta u_z - \frac{2}{a^2} \frac{\partial u_\phi}{\partial \phi} - \frac{2u_z}{a^2} \right) e_z, \end{aligned}$$

where Δf is the Laplacian for scalar functions, and Δu is the Laplace–Beltrami operator on $(0, 2\pi) \times (a, a + h)$ with the product metric. Based on the thermodynamics, the heat source Q is a function of the humidity q . For simplicity, we take the linear approximation

$$Q = \alpha_0 + \alpha_1 q, \quad (2.8)$$

and the humidity source is taken as zero

$$S = 0. \quad (2.9)$$

The boundary conditions are periodic in the ϕ -direction

$$(u, T, q)(\phi + 2\pi, z) = (u, T, q)(\phi, z), \quad (2.10)$$

and free-slip on the Earth surface $z = a$ and the tropopause $z = a + h$,

$$\left. \begin{aligned} u_z = 0, \quad \frac{\partial u_\phi}{\partial z} = 0, \quad T = T_0, \quad q = q_0, \quad \text{at } z = a \\ \text{and } u_z = 0, \quad \frac{\partial u_\phi}{\partial z} = 0, \quad T = T_1, \quad q = 0, \quad \text{at } z = a + h, \end{aligned} \right\} \quad (2.11)$$

where T_0 , T_1 , q_0 and q_1 are constants satisfying

$$T_0 > T_1 \quad \text{and} \quad q_0 > 0. \quad (2.12)$$

The problem (2.7)–(2.11) possesses a steady-state solution

$$\left. \begin{aligned} \tilde{u} &= 0, \\ \tilde{T} &= \frac{\gamma_1}{6\kappa_T} z^3 - \frac{\gamma_0}{2\kappa_T} z^2 + c_1 z + c_0, \\ \tilde{q} &= -\frac{q_0}{h} z + \frac{q_0}{h} (h + a) \\ \tilde{p} &= -\int_0^z \rho_0 g (1 - \alpha_T (\tilde{T} - T_0) - \alpha_q (\tilde{q} - q_0)) \, dz, \end{aligned} \right\} \quad (2.13)$$

and

where

$$\begin{aligned} \gamma_0 &= \alpha_0 + \frac{\alpha_1 q_0 (h + a)}{h}, \quad \gamma_1 = \frac{\alpha_1 q_0}{h}, \\ c_0 &= \frac{a + h}{h} \left(-\frac{\gamma_1}{6\kappa_T} a^3 + \frac{\gamma_0}{2\kappa_T} a^2 + T_0 \right) - \frac{a}{h} \left(-\frac{\gamma_1}{6\kappa_T} (a + h)^3 + \frac{\gamma_0}{2\kappa_T} (a + h)^2 + T_1 \right) \\ \text{and} \quad c_1 &= -\frac{1}{h} (T_0 - T_1) - \frac{1}{\kappa_T} \left(\frac{\gamma_1}{6} (h^2 + 3ha + 3a^2) - \frac{\gamma_0}{2} (h + 2a) \right). \end{aligned}$$

To obtain the non-dimensional form, let

$$\begin{aligned} x &= hx', \quad a = hr_0, \\ t &= \frac{h^2 t'}{\kappa_T}, \quad u = \frac{\kappa_T u'}{h}, \\ T &= (T_0 - T_1) T' + \tilde{T}, \quad q = q_0 q' + \tilde{q} \\ \text{and} \quad p &= \frac{\rho_0 v \kappa_T p'}{h^2} + \tilde{p}, \end{aligned}$$

and

and consider

$$(x_1, x_2) = (r_0 \phi, z) \quad \text{and} \quad (u_1, u_2) = (u_\phi, u_z).$$

Omitting the primes, equations (2.7) become

$$\left. \begin{aligned} \frac{\partial u_1}{\partial t} &= Pr \left(\Delta u_1 + \frac{2}{r_0} \frac{\partial u_2}{\partial x_1} - \frac{1}{r_0^2} u_1 - \frac{\partial p}{\partial x_1} \right) \\ &\quad - \omega u_2 - (u \cdot \nabla) u_1 - \frac{1}{r_0} u_1 u_2, \\ \frac{\partial u_2}{\partial t} &= Pr \left(\Delta u_2 - \frac{2}{r_0} \frac{\partial u_1}{\partial x_1} - \frac{2}{r_0^2} u_2 + RT + \tilde{R}q - \frac{\partial p}{\partial x_2} \right) \\ &\quad + \omega u_1 - (u \cdot \nabla) u_2 + \frac{1}{r_0} u_1^2, \\ \frac{\partial T}{\partial t} &= \Delta T - \frac{1}{T_0 - T_1} \frac{d\tilde{T}(hx_2)}{dx_2} u_2 + \alpha q - (u \cdot \nabla) T, \\ \frac{\partial q}{\partial t} &= Le \Delta q + u_2 - (u \cdot \nabla) q \\ \text{and} \quad \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} &= 0, \end{aligned} \right\} \quad (2.14)$$

and

where the non-dimensional physical parameters are

$$\left. \begin{aligned} Pr &= \frac{\nu}{\kappa_T} \quad \text{the Prandtl number,} \\ R &= \frac{\alpha_T g (T_0 - T_1) h^3}{\kappa_T \nu} \quad \text{the thermal Rayleigh number,} \\ \tilde{R} &= \frac{\alpha_q g q_0 h^3}{\kappa_T \nu} \quad \text{the humidity Rayleigh number,} \\ Le &= \frac{\kappa_q}{\kappa_T} \quad \text{the Lewis number,} \\ \omega &= \frac{2\Omega h^2}{\kappa_T} \quad \text{the Earth rotation} \\ \alpha &= \frac{\alpha_1 q_0 h^2}{\kappa_T (T_0 - T_1)}, \end{aligned} \right\} \quad (2.15)$$

and

and

$$\begin{aligned} -\frac{1}{T_0 - T_1} \frac{d\tilde{T}(hx_2)}{dx_2} &= 1 + \frac{\gamma_0 h^2}{\kappa_T (T_0 - T_1)} \left(x_2 - r_0 - \frac{1}{2} \right) \\ &\quad - \frac{\alpha_1 q_0 h^2}{2\kappa_T (T_0 - T_1)} \left(x_2^2 - r_0^2 - r_0 - \frac{1}{3} \right), \end{aligned} \quad (2.16)$$

where $r_0 < x_2 < r_0 + 1$, and r_0 is the non-dimensional radius of the Earth. For simplicity, we take the average approximation $x_2 = r_0 + \frac{1}{2}$. In this case, (2.16) is given by

$$-\frac{1}{T_0 - T_1} \frac{d\tilde{T}}{dx_2} \Big|_{x_2=r_0+1/2} = 1 + \frac{\alpha_1 q_0 h^2}{24\kappa_T (T_0 - T_1)}.$$

Thus, equation (2.14) is approximated by

$$\left. \begin{aligned} \frac{\partial u_1}{\partial t} &= Pr \left(\Delta u_1 + \frac{2}{r_0} \frac{\partial u_2}{\partial x_1} - \frac{1}{r_0^2} u_1 - \frac{\partial p}{\partial x_1} \right) \\ &\quad - \omega u_2 - (u \cdot \nabla) u_1 - \frac{1}{r_0} u_1 u_2, \\ \frac{\partial u_2}{\partial t} &= Pr \left(\Delta u_2 - \frac{2}{r_0} \frac{\partial u_1}{\partial x_1} - \frac{2}{r_0^2} u_2 + RT + \tilde{R}q - \frac{\partial p}{\partial x_2} \right) \\ &\quad + \omega u_1 - (u \cdot \nabla) u_2 + \frac{1}{r_0} u_1^2, \\ \frac{\partial T}{\partial t} &= \Delta T + \gamma u_2 + \alpha q - (u \cdot \nabla) T, \\ \frac{\partial q}{\partial t} &= Le \Delta q + u_2 - (u \cdot \nabla) q \\ \text{div } u &= 0, \end{aligned} \right\} \quad (2.17)$$

and

where

$$\gamma = 1 + \frac{\alpha}{24}. \quad (2.18)$$

The domain is $M = [0, 2\pi r_0] \times (r_0, r_0 + 1)$, and the boundary conditions are given by

$$\left. \begin{aligned} (u, T, q)(x_1 + 2\pi r_0, x_2) &= (u, T, q)(x_1, x_2) \\ \text{and} \quad u_2 = 0, \quad \frac{\partial u_1}{\partial x_2} &= 0, \quad T = 0, \quad q = 0, \quad \text{at } x_2 = r_0, r_0 + 1. \end{aligned} \right\} \quad (2.19)$$

For the problem (2.17)–(2.19), we set the spaces

$$\left. \begin{aligned} H &= \{(u, T, q) \in L^2(M, \mathbb{R}^4) \mid \operatorname{div} u = 0, u_2 = 0 \text{ at } r_0, r_0 + 1\} \\ \text{and} \quad H_1 &= \{(u, T, q) \in H^2(M, \mathbb{R}^4) \cap H \mid (u, T, q) \text{ satisfies (2.12)}\}. \end{aligned} \right\} \quad (2.20)$$

3. Eigenvalue problem and principle of exchange of stability

(a) Eigenvalue problem

To study the transition of (2.17)–(2.19) from the basic state, we need to consider the following eigenvalue problem:

$$\left. \begin{aligned} Pr \left(\Delta u_1 + \frac{2}{r_0} \frac{\partial u_2}{\partial x_1} - \frac{1}{r_0^2} u_1 - \frac{\partial p}{\partial x_1} \right) - \omega u_2 &= \beta u_1, \\ Pr \left(\Delta u_2 - \frac{2}{r_0} \frac{\partial u_1}{\partial x_1} - \frac{2}{r_0^2} u_2 + RT + \tilde{R}q - \frac{\partial p}{\partial x_2} \right) + \omega u_1 &= \beta u_2, \\ \Delta T + \gamma u_2 + \alpha q &= \beta T, \\ Le \Delta q + u_2 &= \beta q, \\ \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} &= 0 \end{aligned} \right\} \quad (3.1)$$

supplemented with the boundary conditions (2.19). By the boundary conditions (2.19), the eigenvalues of (3.1) can be solved by separation of variables as follows:

$$\psi^1 = \begin{cases} u_1^1 = -u_k(x_2) \sin \frac{kx_1}{r_0}, \\ u_2^1 = v_k(x_2) \cos \frac{kx_1}{r_0}, \\ T^1 = T_k(x_2) \cos \frac{kx_1}{r_0}, \\ q^1 = q_k(x_2) \cos \frac{kx_1}{r_0}, \\ p^1 = A_k(x_2) \cos \frac{kx_1}{r_0} - B_k(x_2) \sin \frac{kx_1}{r_0} \end{cases} \quad (3.2)$$

and

$$\psi^2 = \begin{cases} u_1^2 = u_k(x_2) \cos \frac{kx_1}{r_0}, \\ u_2^2 = v_k(x_2) \sin \frac{kx_1}{r_0}, \\ T^2 = T_k(x_2) \sin \frac{kx_1}{r_0}, \\ q^2 = q_k(x_2) \sin \frac{kx_1}{r_0}, \\ p^2 = A_k(x_2) \sin \frac{kx_1}{r_0} + B_k(x_2) \cos \frac{kx_1}{r_0}, \end{cases} \quad (3.3)$$

for $k = 1, 2, 3, \dots$. Based on the continuity equation in (3.1), we have

$$u_k = \frac{r_0}{k} \frac{dv_k}{dx_2}. \quad (3.4)$$

Let

$$B_k = \frac{\omega r_0}{k Pr} v_k(x_2) \quad \text{and} \quad A_k = p_k + \frac{2}{r_0} v_k(x_2). \quad (3.5)$$

Owing to (2.19), plugging (3.2) or (3.3) into (3.1), we obtain the following system of ordinary differential equations:

$$\left. \begin{aligned} Pr \left(D_k^2 u_k - \frac{1}{r_0^2} u_k - \frac{k}{r_0} p_k \right) &= \beta u_k, \\ Pr \left(D_k^2 v_k - \frac{2}{r_0^2} v_k + RT_k + \tilde{R} q_k - Dp_k \right) &= \beta v_k, \\ D_k^2 T_k + \gamma v_k + \alpha q_k &= \beta T_k, \\ Le D_k^2 q_k + v_k &= \beta q_k, \\ Du_k = 0, \quad v_k = 0, \quad T_k = 0, \quad q_k = 0, \quad &\text{at } x_2 = r_0, r_0 + 1, \end{aligned} \right\} \quad (3.6)$$

and

where

$$D = \frac{d}{dx_2} \quad \text{and} \quad D_k^2 = \frac{d^2}{dx_2^2} - \frac{k^2}{r_0^2}.$$

Plugging $v_k = \sin j\pi(x_2 - r_0)$ into (3.6), we see that the eigenvalue β satisfies the cubic equation

$$\begin{aligned} (\alpha_{kj}^2 + \beta)(Le \alpha_{kj}^2 + \beta) \left(Pr \alpha_{kj}^2 + \frac{Pr}{r_0^2} + \beta \right) \alpha_{kj}^2 + \frac{k^2 Pr}{r_0^4} (\alpha_{kj}^2 + \beta)(Le \alpha_{kj}^2 + \beta) \\ - \frac{k^2 Pr R}{r_0^2} (\gamma Le \alpha_{kj}^2 + \gamma \beta + \alpha) - \frac{k^2 Pr \tilde{R}}{r_0^2} (\alpha_{kj}^2 + \beta) = 0, \end{aligned} \quad (3.7)$$

where

$$\alpha_{kj} = \left(j^2 \pi^2 + \frac{k^2}{r_0^2} \right)^{1/2} \quad (j \geq 1, k \geq 1). \quad (3.8)$$

(b) Principle of exchange of stabilities

The linear stability of the problem (2.17)–(2.19) is dictated precisely by the eigenvalues of (3.1), which are determined by (3.7). The expansion of (3.7) is

$$\begin{aligned} \beta^3 + \left[(1 + Le + Pr) \alpha_{kj}^2 + \frac{Pr}{r_0^2} + \frac{k^2 Pr}{r_0^4 \alpha_{kj}^2} \right] \beta^2 \\ + \left[(Pr + Le Pr + Le) \alpha_{kj}^4 + \frac{Pr}{r_0^2} (1 + Le) \left(\alpha_{kj}^2 + \frac{k^2}{r_0^2} \right) - \frac{Pr k^2}{r_0^2 \alpha_{kj}^2} (\gamma R + \tilde{R}) \right] \beta \\ + Le \alpha_{kj}^4 \left(Pr \alpha_{kj}^2 + \frac{Pr}{r_0^2} \right) + \frac{Pr k^2}{r_0^2 \alpha_{kj}^2} \left(-\alpha_{kj}^2 \tilde{R} - \alpha R - \gamma Le \alpha_{kj}^2 R + \frac{Le \alpha_{kj}^4}{r_0^2} \right) \\ = 0. \end{aligned} \quad (3.9)$$

We see that $\beta = 0$ is a solution of (3.9) if and only if

$$Le \alpha_{kj}^4 \left(Pr \alpha_{kj}^2 + \frac{Pr}{r_0^2} \right) + \frac{Pr k^2}{r_0^2 \alpha_{kj}^2} \left(-\alpha_{kj}^2 \tilde{R} - \alpha R - \gamma Le \alpha_{kj}^2 R + \frac{Le \alpha_{kj}^4}{r_0^2} \right) = 0. \quad (3.10)$$

Inferring from (2.15) and (2.18),

$$\left. \begin{aligned} \alpha R &= \frac{\alpha_1 \alpha_T g q_0 h^5}{\kappa_T^2 \nu} = \frac{\alpha_1 \alpha_T h^2}{\alpha_q \kappa_T} \tilde{R} \\ \gamma R &= R + \frac{\alpha_1 \alpha_T g q_0 h^5}{24 \kappa_T^2 \nu} = R + \frac{\alpha_1 \alpha_T h^2}{24 \alpha_q \kappa_T} \tilde{R}. \end{aligned} \right\} \quad (3.11)$$

and

Hence, we rewrite (3.10) as

$$R = - \left(\frac{1}{Le} + \left(1 + \frac{Le \alpha_{kj}^2}{24} \right) \frac{\alpha_1 \alpha_T h^2}{Le \alpha_q \alpha_{kj}^2 \kappa_T} \right) \tilde{R} + \alpha_{kj}^4 \frac{r_0^2}{k^2} \left(\alpha_{kj}^2 + \frac{1}{r_0^2} \right) + \frac{\alpha_{kj}^2}{r_0^2}. \quad (3.12)$$

We define the critical Rayleigh number R_c as

$$\begin{aligned} R_c &= \min_{k,j \geq 1} \left[- \left(\frac{1}{Le} + \left(1 + \frac{Le \alpha_{kj}^2}{24} \right) \frac{\alpha_1 \alpha_T h^2}{Le \alpha_q \alpha_{kj}^2 \kappa_T} \right) \tilde{R} + \alpha_{kj}^4 \frac{r_0^2}{k^2} \left(\alpha_{kj}^2 + \frac{1}{r_0^2} \right) + \frac{\alpha_{kj}^2}{r_0^2} \right] \\ &= \min_{k \geq 1} \left[- \left(\frac{1}{Le} + \left(1 + \frac{Le \alpha_{k1}^2}{24} \right) \frac{\alpha_1 \alpha_T h^2}{Le \alpha_q \alpha_{k1}^2 \kappa_T} \right) \tilde{R} + \alpha_{k1}^4 \frac{r_0^2}{k^2} \left(\alpha_{k1}^2 + \frac{1}{r_0^2} \right) + \frac{\alpha_{k1}^2}{r_0^2} \right], \end{aligned} \quad (3.13)$$

where

$$\alpha_{k1}^2 = \pi^2 + \frac{k^2}{r_0^2}.$$

Based on (3.9)–(3.12), by definition (3.13), we shall prove the following principle of exchange of stabilities later.

Lemma 3.1. Let β_{jk} be the eigenvalues of (3.1) that satisfy (3.9). Let $k_c \geq 1$ be integer minimizing (3.13), and $\beta_{k_c}^i$ ($1 \leq i \leq 3$) be solution of (3.9) with $(k, j) = (k_c, 1)$, and

$$\Re \beta_{k_c}^3 \leq \Re \beta_{k_c}^2 \leq \Re \beta_{k_c}^1.$$

Then $\beta_{k_c}^1$ must be real near $R = R_c$, and

$$\beta_{k_c}^1(R) \begin{cases} < 0, & R < R_c, \\ = 0, & R = R_c, \\ > 0, & R > R_c, \end{cases}$$

$$\Re \beta_{k_c}^j(R_c) < 0, \quad \text{for } j = 2, 3.$$

Moreover, we have

$$\beta_{jk}(R_c) < 0 \quad \text{for } \beta_{jk} \text{ being real and } \beta_{jk} \neq \beta_{k_c}^1.$$

Remark 3.2. The value R_c defined by (3.13) is called the critical Rayleigh number at which the principle of exchange of stability (PES) holds. It provides the critical temperature difference $\Delta T_c = T_0 - T_1$ given by

$$\frac{\alpha_T g h^3 \Delta T_c}{\kappa_T \nu} = R_c.$$

Equivalently, we have

$$\begin{aligned} \Delta T_c &= \frac{\kappa_T \nu}{\alpha_T g h^3} \left[- \left(\frac{1}{Le} + \left(1 + \frac{Le \alpha_{k_c 1}^2}{24} \right) \frac{\alpha_1 \alpha_T h^2}{Le \alpha_q \alpha_{k_c 1}^2 \kappa_T} \right) \tilde{R} \right. \\ &\quad \left. + \alpha_{k_c 1}^4 \frac{r_0^2}{k_c^2} \left(\alpha_{k_c 1}^2 + \frac{1}{r_0^2} \right) + \frac{\alpha_{k_c 1}^2}{r_0^2} \right], \end{aligned} \quad (3.14)$$

where $k_c \geq 1$ is the integer which satisfies (3.13).

Next, we consider the PES for the complex eigenvalues of (3.1). Let $\beta = i\rho_0$ ($\rho_0 \neq 0$) be a zero of (3.9). Then

$$\rho_0^2 = (Pr + Le \ Pr + Le)\alpha_{kj}^4 + \frac{Pr}{r_0^2}(1 + Le)\left(\alpha_{kj}^2 + \frac{k^2}{r_0^2}\right) - \frac{Pr k^2}{r_0^2 \alpha_{kj}^2}(\gamma R + \tilde{R})$$

and

$$\rho_0^2 = \left[Le \ \alpha_{kj}^4 \left(Pr \ \alpha_{kj}^2 + \frac{Pr}{r_0^2} \right) + \frac{Pr k^2}{r_0^2 \alpha_{kj}^2} \left(-\alpha_{kj}^2 \tilde{R} - \alpha R - \gamma Le \ \alpha_{kj}^2 R + \frac{Le \ \alpha_{kj}^4}{r_0^2} \right) \right] \Bigg/ \left[(1 + Le + Pr)\alpha_{kj}^2 + \frac{Pr}{r_0^2} \right].$$

Hence, equation (3.9) has a pair of purely imaginary solutions $\pm i\rho_0$ if and only if the following condition holds true:

$$\begin{aligned} & \left[(1 + Le + Pr)\alpha_{kj}^2 + \frac{Pr}{r_0^2} \right] \left[(Pr + Le + Le \ Pr)\alpha_{kj}^4 \right. \\ & \quad \left. + \frac{Pr}{r_0^2}(1 + Le)\left(\alpha_{kj}^2 + \frac{k^2}{r_0^2}\right) - \frac{Pr k^2}{r_0^2 \alpha_{kj}^2}(\gamma R + \tilde{R}) \right] \\ & = \left[Le \ \alpha_{kj}^4 \left(Pr \ \alpha_{kj}^2 + \frac{2Pr}{r_0^2} \right) + \frac{Pr k^2}{r_0^2 \alpha_{kj}^2} \left(-\alpha_{kj}^2 \tilde{R} - \alpha R - \gamma Le \ \alpha_{kj}^2 R + \frac{Le \ \alpha_{kj}^4}{r_0^2} \right) \right] \\ & > 0. \end{aligned} \quad (3.15)$$

It follows from (3.11) and (3.15) that

$$\begin{aligned} \left(\frac{Pr}{r_0^2} + (Pr + 1)\alpha_{kj}^2 \right) R & = \left[\frac{\alpha_1 \alpha_T h^2}{\alpha_q \kappa_T} \left(1 - \frac{1}{24} \left(\frac{Pr}{r_0^2} + (Pr + 1)\alpha_{k_c}^2 \right) \right) - \left(\frac{Pr}{r_0^2} + (Pr + Le)\alpha_{k_c}^2 \right) \right] \tilde{R} \\ & \quad + \frac{r_0^2 \alpha_{kj}^2}{k^2 Pr} \left(Le(1 + Le)\alpha_{kj}^6 + Pr \left[(1 + Le + Pr)\alpha_{kj}^2 + \frac{Pr}{r_0^2} \right] \left[(1 + Le)\alpha_{kj}^4 \right. \right. \\ & \quad \left. \left. + \frac{(1 + Le)}{r_0^2} \alpha_{kj}^2 + \frac{k^2 Le}{r_0^4} \right] + \frac{Pr k^2}{r_0^4} \left[(1 + Pr)\alpha_{kj}^2 + \frac{Pr}{r_0^2} \right] \right), \end{aligned} \quad (3.16)$$

where $\alpha_{k_c}^2$ is as defined in (3.8).

According to (3.16), we define the critical Rayleigh number for the complex PES as follows:

$$\begin{aligned} R_c^* & = \min_{k,j \geq 1} \frac{1}{Pr/r_0^2 + (Pr + 1)\alpha_{kj}^2} \\ & \quad \times \left\{ \left[\frac{\alpha_1 \alpha_T h^2}{\alpha_q \kappa_T} \left(1 - \frac{1}{24} \left(\frac{Pr}{r_0^2} + (Pr + 1)\alpha_{k_c}^2 \right) \right) - \left(\frac{Pr}{r_0^2} + (Pr + Le)\alpha_{k_c}^2 \right) \right] \tilde{R} \right. \\ & \quad + \frac{r_0^2 \alpha_{kj}^2}{k^2 Pr} \left(Le(1 + Le)\alpha_{kj}^6 + Pr \left[(1 + Le + Pr)\alpha_{kj}^2 + \frac{Pr}{r_0^2} \right] \left[(1 + Le)\alpha_{kj}^4 \right. \right. \\ & \quad \left. \left. + \frac{(1 + Le)}{r_0^2} \alpha_{kj}^2 + \frac{k^2 Le}{r_0^4} \right] + \frac{Pr k^2}{r_0^4} \left[(1 + Pr)\alpha_{kj}^2 + \frac{Pr}{r_0^2} \right] \right) \Bigg\}. \end{aligned} \quad (3.17)$$

Then we have the following complex PES.

Lemma 3.3. Let k_c^*, j_c^* be the integers satisfying (3.17), and $\beta_{k_c^* j_c^*}^1$ and $\beta_{k_c^* j_c^*}^2$ be the pair of complex eigenvalues of (3.9) with $(k, j) = (k_c^*, j_c^*)$ near $R = R_c^*$. Then,

$$\Re \beta_{k_c^* j_c^*}^1 = \Re \beta_{k_c^* j_c^*}^2 \begin{cases} < 0, & R < R_c^*, \\ = 0, & R = R_c^*, \\ > 0, & R > R_c^*. \end{cases}$$

Furthermore, for all complex eigenvalues β_{kj} of (3.1), we have

$$\Re \beta_{kj}(R_c^*) < 0, \quad \text{for } (k, j) \neq (k_c^*, j_c^*).$$

(c) Proof of lemmas 3.1 and 3.3

We note that all eigenvalues of (3.1) are determined by (3.9) and the eigenvalue equations

$$\begin{cases} \Delta T + \alpha q = \beta T \\ Le \Delta q = \beta q, \end{cases} \quad (3.18)$$

and

with the boundary condition (2.19). It is clear that the eigenvalues of (3.18) are real and negative. Hence, it suffices to prove lemmas 3.1 and 3.3 for eigenvalues β_{kj} of (3.9). We rewrite (3.9) as

$$\beta^3 + b_2 \beta^2 + b_1 \beta + b_0 = 0, \quad (3.19)$$

where

$$\begin{aligned} b_2 &= (1 + Le + Pr) \alpha_{kj}^2 + \frac{Pr}{r_0^2} + \frac{k^2 Pr}{r_0^4 \alpha_{kj}^2}, \\ b_1 &= (Pr + Le Pr + Le) \alpha_{kj}^4 + \frac{Pr}{r_0^2} (1 + Le) \left(\alpha_{kj}^2 + \frac{k^2}{r_0^2} \right) - \frac{Pr k^2}{r_0^2 \alpha_{kj}^2} (\gamma R + \tilde{R}) \\ \text{and} \quad b_0 &= Le \alpha_{kj}^4 \left(Pr \alpha_{kj}^2 + \frac{Pr}{r_0^2} \right) + \frac{Pr k^2}{r_0^2 \alpha_{kj}^2} \left(-\alpha_{kj}^2 \tilde{R} - \alpha R - \gamma Le \alpha_{kj}^2 R + \frac{Le \alpha_{kj}^4}{r_0^2} \right). \end{aligned}$$

We see that

$$b_0, b_1, b_2 > 0 \quad \text{at } R = 0 \text{ for all } k, j \geq 1, \quad (3.20)$$

which implies that all real eigenvalues of (3.9) are negative at $R = 0$. By the definition of R_c , we find

$$b_0 > 0, \quad \forall (k, j) \neq (k_c, 1), \quad 0 \leq R \leq R_c \quad \left\{ \begin{array}{l} > 0, \quad R < R_c \\ = 0, \quad R = R_c, \\ < 0, \quad R > R_c, \end{array} \right. \quad \text{for } (k, j) = (k_c, 1). \quad (3.21)$$

As $b_0 = -\beta_{kj}^1 \beta_{kj}^2 \beta_{kj}^3$, lemma 3.1 follows from (3.20) and (3.21).

Next, let $\beta_{k_c^* j_c^*}$, the solution of (3.9), take the form

$$\begin{cases} \beta_{k_c^* j_c^*}(R) = \lambda(R) + i\sigma(R) \\ \lambda(R) \rightarrow 0, \quad \sigma(R) \rightarrow \sigma_0, \quad \text{as } R \rightarrow R_c^*. \end{cases} \quad (3.22)$$

and

Inserting (3.22) into (3.19), we obtain

$$\begin{cases} (-3\sigma^2 + b_1)\lambda + b_0 - b_2\sigma^2 + o(\lambda) = 0 \\ -\sigma^3 + \sigma b_1 + 2\sigma b_2\lambda + o(\lambda) = 0. \end{cases} \quad (3.23)$$

and

As $\sigma_0 \neq 0$, we infer from (3.23) that

$$\lambda(R) + o(\lambda) = \frac{b_0 - b_2\sigma^2}{3\sigma^2 - b_1} = \frac{b_0 - b_1b_2 - 2b_2^2\lambda}{2b_1 + 6b_2\lambda} + o(\lambda).$$

Thus, we have

$$\lambda(R) = \frac{b_0 - b_1b_2 - 2b_2^2\lambda}{2b_1 + 6b_2\lambda} + o(\lambda). \quad (3.24)$$

Under the condition (3.15), we see $b_1 > 0$. It follows from (3.22) and (3.24) that

$$\Re \epsilon \beta_{k_c^* j_c^*}(R) = \lambda(R) \begin{cases} < 0, & b_0 < b_1b_2, \\ = 0, & b_0 = b_1b_2, \\ > 0, & b_0 > b_1b_2, \end{cases} \quad (3.25)$$

for R near R_c^* . By the definition of R_c^* , it is easy to see that

$$b_0 \begin{cases} < b_1b_2, & R < R_c^*, \\ = b_1b_2, & R = R_c^*, \\ > b_1b_2, & R > R_c^*. \end{cases} \quad (3.26)$$

This proves lemma 3.3.

By lemmas 3.1 and 3.3, we immediately obtain the following theorem which provides a criterion to determine the equilibrium and the spatio-temporal oscillation transitions.

Theorem 3.4. *Let R_c and R_c^* be the parameters defined by (3.13) and (3.17) respectively. Then, the following assertions hold true.*

- (i) *When $R_c < R_c^*$, the first critical-crossing eigenvalue of the problem (3.1) is $\beta_{k_c^*}^1$ given by lemma 3.1, i.e.*

$$\beta_{k_c^*}^1(R) \begin{cases} < 0, & \text{if } R < R_c, \\ = 0, & \text{if } R = R_c, \\ > 0, & \text{if } R > R_c \end{cases} \quad (3.27)$$

and

$$\Re \epsilon \beta(R_c) < 0, \quad (3.28)$$

for all other eigenvalues β of (3.1).

- (ii) *When $R_c^* < R_c$, the first critical-crossing eigenvalues are the pair of complex eigenvalues $\beta_{k_c^* j_c^*}^1$ and $\beta_{k_c^* j_c^*}^2$ given by lemma 3.3, namely,*

$$\Re \epsilon \beta_{k_c^* j_c^*}^1(R) = \Re \epsilon \beta_{k_c^* j_c^*}^2(R) \begin{cases} < 0, & \text{if } R < R_c^*, \\ = 0, & \text{if } R = R_c^*, \\ > 0, & \text{if } R > R_c^* \end{cases} \quad (3.29)$$

and

$$\Re \epsilon \beta(R_c^*) < 0, \quad (3.30)$$

for all other eigenvalues $\beta(R)$ of (3.1).

Remark 3.5. In the atmospheric science, the integer j_c^* in (3.29) is 1. Hence, the critical Rayleigh numbers R_c and R_c^* are given by

$$R_c = - \left(\frac{1}{Le} + \left(1 + \frac{Le \alpha_{k_c}^2}{24} \right) \frac{\alpha_1 \alpha_T h^2}{Le \alpha_q \alpha_{k_c}^2 \kappa_T} \right) \tilde{R} + \alpha_{k_c}^4 \frac{r_0^2}{k^2} \left(\alpha_{k_c}^2 + \frac{1}{r_0^2} \right) + \frac{\alpha_{k_c}^2}{r_0^2} \quad (3.31)$$

and

$$\begin{aligned} R_c^* = & \frac{1}{Pr/r_0^2 + (Pr+1)\alpha_{k_c}^2} \\ & \times \left\{ \left[\frac{\alpha_1 \alpha_T h^2}{\alpha_q \kappa_T} \left(1 - \frac{1}{24} \left(\frac{Pr}{r_0^2} + (Pr+1)\alpha_{k_c}^2 \right) \right) - \left(\frac{Pr}{r_0^2} + (Pr+Le)\alpha_{k_c}^2 \right) \right] \tilde{R} \right. \\ & + \frac{r_0^2 \alpha_{k_c}^2}{k^2 Pr} \left(Le(1+Le)\alpha_{k_c}^6 + Pr \left[(1+Le+Pr)\alpha_{k_c}^2 + \frac{Pr}{r_0^2} \right] \left[(1+Le)\alpha_{k_c}^4 \right. \right. \\ & \left. \left. + \frac{(1+Le)}{r_0^2} \alpha_{k_c}^2 + \frac{k_c^{*2} Le}{r_0^4} \right] + \frac{Pr k_c^{*2}}{r_0^4} \left[(1+Pr)\alpha_{k_c}^2 + \frac{Pr}{r_0^2} \right] \right) \left. \right\}, \quad (3.32) \end{aligned}$$

where

$$\alpha_{k_c}^2 = \pi^2 + \frac{k_c^2}{r_0^2} \quad \text{and} \quad \alpha_{k_c}^{*2} = \pi^2 + \frac{k_c^{*2}}{r_0^2}. \quad (3.33)$$

4. Transition theorem

Inferring from theorem 4.1, system (2.17)–(2.19) has a transition to equilibria at $R = R_c$ provided $R_c < R_c^*$ and has a transition to spatio-temporal oscillation at $R = R_c^*$ provided $R_c^* < R_c$, where the critical values R_c and R_c^* are defined by (3.13) and (3.17), respectively.

Theorem 4.1. *For the problem (2.17)–(2.19), we have the following assertions.*

- (1) When $R < \min\{R_c, R_c^*\}$, the equilibrium solution $(u, T, q) = 0$ is stable in H .
- (2) If $R_c < R_c^*$, then this problem has a continuous transition at $R = R_c$, and bifurcates from $((u, T, q), R) = (0, R_c)$ to an attractor $\Sigma_R = S^1$ on $R > R_c$ which is a cycle of steady-state solutions.
- (3) As $R_c^* < R_c$, the problem has a transition at $R = R_c^*$, which is either of continuous type or of jump type, and it transits to a spatio-temporal oscillation solution. In particular, if the transition is continuous, then there is an attractor of three-dimensional homological sphere S^3 is bifurcated from $((u, T, q), R) = (0, R_c^*)$ on $R > R_c^*$, which contains no steady-state solutions.

Remark 4.2. Although the introduction of the humidity effect leads to substantial difficulty from the mathematical point of view, we see from theorem 4.1 that the humidity does not affect the type of dynamic transition the system undergoes. In particular, as Ma & Wang [17,18], the random transition behaviour of the system in the general case explains general features of the observed abrupt changes between strong El Nino and strong La Nina states. Also, with the deterministic model considered in this article, the randomness is closely related to the uncertainty/fluctuations of the initial data between the narrow basins of attractions of the corresponding metastable events, and the deterministic feature is represented by a deterministic coupled atmospheric and oceanic model predicting the basins of attraction and the SST. From the predictability and prediction point of view, it is crucial to capture more detailed information on the delay feedback mechanism of the SST. For this purpose, the study of an explicit multi-scale coupling mechanism to the ocean is inevitable. In fact, the new mechanism strongly suggests the need and importance of the coupled ocean–atmosphere models for ENSO predication in [1,3–11]. Finally, by (3.31), the critical thermal Rayleigh number is slightly smaller than that in the case without moisture factor [17,18].

Proof of theorem 4.1. We shall prove this theorem with several steps.

Step 1. Let H and H_1 be the spaces defined by (2.20). We define the operators $L_R = A + B_R : H_1 \rightarrow H$ and $G : H_1 \rightarrow H$ by

$$\left. \begin{aligned} A\psi &= P \left(Pr \left(\Delta u_1 + \frac{2}{r_0} \frac{\partial u_2}{\partial x_1} \right), Pr \left(\Delta u_2 - \frac{2}{r_0} \frac{\partial u_1}{\partial x_1} \right), \Delta T, Le \Delta q \right), \\ B_R\psi &= P \left(-\frac{Pr}{r_0^2} u_1 - \omega u_2, -Pr \left(\frac{2}{r_0^2} u_2 - RT - \tilde{R}q \right) + \omega u_1, \gamma u_2 + \alpha q, u_2 \right), \\ \text{and} \quad G(\psi) &= -P \left((u \cdot \nabla) u_1 + \frac{1}{r_0} u_1 u_2, (u \cdot \nabla) u_2 - \frac{1}{r_0} u_1^2, (u \cdot \nabla) T, (u \cdot \nabla) q \right), \end{aligned} \right\} \quad (4.1)$$

where $P : L^2(M, \mathbb{R}^4) \rightarrow H$ is the Leray projection and

$$\psi = (u, T, q) \in H_1.$$

Thus, the problem (2.17)–(2.19) is expressed in form of

$$\frac{d\psi}{dt} = L_R \psi + G(\psi), \quad (4.2)$$

and the eigenvalue problem (3.1) with the condition (2.19) is rewritten as

$$L_R \psi = \beta \psi. \quad (4.3)$$

Step 2. We shall calculate the centre manifold reduction for (4.2) in this step. Let $\psi_{k_c}^1$ and $\psi_{k_c}^2$ be the eigenfunctions of (4.3) corresponding to $\beta_{k_c}^1(R)$, where $\beta_{k_c}^1$ is the eigenvalue of (4.3) in the case of (3.27). Denote the conjugate eigenfunctions of $\psi_{k_c}^1$ and $\psi_{k_c}^2$ by $\psi_{k_c}^{1*}$ and $\psi_{k_c}^{2*}$, i.e.

$$L_R^* \psi_{k_c}^{i*} = \beta_{k_c}^1 \psi_{k_c}^{i*}, \quad i = 1, 2. \quad (4.4)$$

The corresponding equations of (4.4) read

$$\left. \begin{aligned} Pr \left(\Delta u_1^* + \frac{2}{r_0} \frac{\partial u_2^*}{\partial x_1} - \frac{1}{r_0^2} u_1^* - \frac{\partial p^*}{\partial x_1} \right) + \omega u_2^* &= \beta u_1^*, \\ Pr \left(\Delta u_2^* - \frac{2}{r_0} \frac{\partial u_1^*}{\partial x_1} - \frac{2}{r_0^2} u_2^* - \frac{\partial p^*}{\partial x_2} \right) + \gamma T^* + q^* - \omega u_1^* &= \beta u_2^*, \\ \Delta T^* + Pr Ru_2^* &= \beta T^*, \\ Le \Delta q^* + Pr \tilde{R} u_2^* + \alpha T^* &= \beta q^*, \\ \frac{\partial u_1^*}{\partial x_1} + \frac{\partial u_2^*}{\partial x_2} &= 0. \end{aligned} \right\} \quad (4.5)$$

Express $\psi \in H_1$ as

$$\psi = x \psi_{k_c}^1 + y \psi_{k_c}^2 + \Phi(x, y, R), \quad (4.6)$$

where $(x, y) \in \mathbb{R}^2$ and $\Phi(x, y, R)$ is the centre manifold function of (4.2) near $R = R_c$. For the reference of centre manifold functions, see, for example, Section 2.4 of [23]. Then, the reduction equations of (4.2) are

$$\left. \begin{aligned} \frac{dx}{dt} &= \beta_{k_c}^1 x + \frac{1}{\langle \psi_{k_c}^1, \psi_{k_c}^{1*} \rangle} \langle G(\psi), \psi_{k_c}^{1*} \rangle \\ \frac{dy}{dt} &= \beta_{k_c}^1 y + \frac{1}{\langle \psi_{k_c}^2, \psi_{k_c}^{2*} \rangle} \langle G(\psi), \psi_{k_c}^{2*} \rangle. \end{aligned} \right\} \quad (4.7)$$

Step 3. Computation of $\psi_{k_c}^i$ and $\psi_{k_c}^{i*}$ ($i = 1, 2$). Based on (3.2)–(3.8), we can derive

$$\psi_{k_c}^1 = \begin{cases} u_1^1 = -u_{k_c} \cos \pi(x_2 - r_0) \sin \frac{k_c x_1}{r_0}, \\ u_2^1 = v_{k_c} \sin \pi(x_2 - r_0) \cos \frac{k_c x_1}{r_0}, \\ T_1^1 = T_{k_c} \sin \pi(x_2 - r_0) \cos \frac{k_c x_1}{r_0}, \\ q^1 = q_{k_c} \sin \pi(x_2 - r_0) \cos \frac{k_c x_1}{r_0} \end{cases} \quad (4.8)$$

and

$$\psi_{k_c}^2 = \begin{cases} u_1^2 = u_{k_c} \cos \pi(x_2 - r_0) \cos \frac{k_c x_1}{r_0}, \\ u_2^2 = v_{k_c} \sin \pi(x_2 - r_0) \sin \frac{k_c x_1}{r_0}, \\ T^2 = T_{k_c} \sin \pi(x_2 - r_0) \sin \frac{k_c x_1}{r_0}, \\ q^2 = q_{k_c} \sin \pi(x_2 - r_0) \sin \frac{k_c x_1}{r_0}, \end{cases} \quad (4.9)$$

where

$$\left. \begin{aligned} u_{k_c} &= \frac{r_0 \pi}{k_c}, \quad v_{k_c} = 1, \\ T_{k_c} &= \frac{\alpha + \gamma(Le \alpha_{k_c}^2 + \beta_{k_c}^1)}{(\alpha_{k_c}^2 + \beta_{k_c}^1)(Le \alpha_{k_c}^2 + \beta_{k_c}^1)}, \quad q_{k_c} = \frac{1}{Le \alpha_{k_c}^2 + \beta_{k_c}^1}. \end{aligned} \right\} \quad (4.10)$$

and

Similarly, we can also derive from (4.5) the conjugate eigenfunctions as follows:

$$\psi_{k_c}^{1*} = \begin{cases} u_1^{1*} = -u_{k_c}^* \cos \pi(x_2 - r_0) \sin \frac{k_c x_1}{r_0}, \\ u_2^{1*} = v_{k_c}^* \sin \pi(x_2 - r_0) \cos \frac{k_c x_1}{r_0}, \\ T_1^{1*} = T_{k_c}^* \sin \pi(x_2 - r_0) \cos \frac{k_c x_1}{r_0}, \\ q^{1*} = q_{k_c}^* \sin \pi(x_2 - r_0) \cos \frac{k_c x_1}{r_0} \end{cases} \quad (4.11)$$

and

$$\psi_{k_c}^{2*} = \begin{cases} u_1^{2*} = u_{k_c}^* \cos \pi(x_2 - r_0) \cos \frac{k_c x_1}{r_0}, \\ u_2^{2*} = v_{k_c}^* \sin \pi(x_2 - r_0) \sin \frac{k_c x_1}{r_0}, \\ T^{2*} = T_{k_c}^* \sin \pi(x_2 - r_0) \sin \frac{k_c x_1}{r_0}, \\ q^{2*} = q_{k_c}^* \sin \pi(x_2 - r_0) \sin \frac{k_c x_1}{r_0}, \end{cases} \quad (4.12)$$

where

$$\left. \begin{aligned} u_{k_c}^* &= \frac{r_0 \pi}{k_c}, \quad v_{k_c}^* = 1, \\ T_{k_c}^* &= \frac{Pr R}{\alpha_{k_c}^2 + \beta_{k_c}^1}, \quad q_{k_c}^* = \frac{\alpha Pr R - Pr \tilde{R}(\alpha_{k_c}^2 + \beta_{k_c}^1)}{(\alpha_{k_c}^2 + \beta_{k_c}^1)(Le \alpha_{k_c}^2 + \beta_{k_c}^1)}. \end{aligned} \right\} \quad (4.13)$$

and

Step 4. Computation of the second-order approximation of the centre manifold function. The nonlinear operator G defined in (4.1) is bilinear, which can be expressed as

$$G(\psi, \phi) = -P \begin{pmatrix} u_1 \frac{\partial v_1}{\partial x_1} + u_2 \frac{\partial v_1}{\partial x_2} + \frac{1}{r_0} u_1 v_2 \\ u_1 \frac{\partial v_2}{\partial x_1} + u_2 \frac{\partial v_2}{\partial x_2} - \frac{1}{r_0} u_1 v_1 \\ u_1 \frac{\partial \tilde{T}}{\partial x_1} + u_2 \frac{\partial \tilde{T}}{\partial x_2} \\ u_1 \frac{\partial \tilde{q}}{\partial x_1} + u_2 \frac{\partial \tilde{q}}{\partial x_2} \end{pmatrix}, \quad (4.14)$$

where $\psi = (u_1, u_2, T, q)$ and $\phi = (v_1, v_2, \tilde{T}, \tilde{q})$. Direct calculation shows

$$\begin{aligned} G(\psi_k^1, \psi_k^1) = & -P \left[\left(\frac{r_0 \pi^2}{2k} \sin \frac{2kx_1}{r_0}, \frac{\pi}{2} \sin 2\pi(x_2 - r_0) - \frac{r_0 \pi^2}{4k^2} [1 + \cos 2\pi(x_2 - r_0)], 0, 0 \right)^t \right. \\ & + \left(-\frac{\pi}{4k} \sin 2\pi(x_2 - r_0) \sin \frac{2kx_1}{r_0}, \frac{r_0 \pi^2}{4k^2} \cos 2\pi(x_2 - r_0) \cos \frac{2kx_1}{r_0}, 0, 0 \right)^t \\ & \left. + \left(0, \frac{r_0 \pi^2}{4k^2} \cos \frac{2kx_1}{r_0}, \frac{T_k \pi}{2} \sin 2\pi(x_2 - r_0), \frac{q_k \pi}{2} \sin 2\pi(x_2 - r_0) \right)^t \right]. \end{aligned} \quad (4.15)$$

As the first two terms on the right-hand side of (4.15) are gradient fields, we obtain

$$G(\psi_k^1, \psi_k^1) = \left(0, -\frac{r_0 \pi^2}{4k^2} \cos \frac{2kx_1}{r_0}, -\frac{T_k \pi}{2} \sin 2\pi(x_2 - r_0), -\frac{q_k \pi}{2} \sin 2\pi(x_2 - r_0) \right)^t. \quad (4.16)$$

Similarly, we can derive

$$\begin{aligned} G(\psi_k^1, \psi_k^2) &= \left(-\frac{r_0 \pi^2}{2k} \cos 2\pi(x_2 - r_0) + \frac{\pi}{4k} \sin 2\pi(x_2 - r_0), -\frac{r_0 \pi^2}{4k^2} \sin \frac{2kx_1}{r_0}, 0, 0 \right)^t, \\ G(\psi_k^2, \psi_k^1) &= \left(\frac{r_0 \pi^2}{2k} \cos 2\pi(x_2 - r_0) - \frac{\pi}{4k} \sin 2\pi(x_2 - r_0), -\frac{r_0 \pi^2}{4k^2} \sin \frac{2kx_1}{r_0}, 0, 0 \right)^t, \\ \text{and } G(\psi_k^2, \psi_k^2) &= \left(0, \frac{r_0 \pi^2}{4k^2} \cos \frac{2kx_1}{r_0}, -\frac{T_k \pi}{2} \sin 2\pi(x_2 - r_0), -\frac{q_k \pi}{2} \sin 2\pi(x_2 - r_0) \right)^t \end{aligned} \quad (4.17)$$

and

$$\begin{aligned} G(\psi_k^1, \psi_k^{1*}) &= \left(0, -\frac{r_0 \pi^2}{4k^2} \cos \frac{2kx_1}{r_0}, -\frac{T_k^* \pi}{2} \sin 2\pi(x_2 - r_0), -\frac{q_k^* \pi}{2} \sin 2\pi(x_2 - r_0) \right)^t, \\ G(\psi_k^1, \psi_k^{2*}) &= \left(-\frac{r_0 \pi^2}{2k} \cos 2\pi(x_2 - r_0) + \frac{\pi}{4k} \sin 2\pi(x_2 - r_0), -\frac{r_0 \pi^2}{4k^2} \sin \frac{2kx_1}{r_0}, 0, 0 \right)^t, \\ G(\psi_k^2, \psi_k^{1*}) &= \left(\frac{r_0 \pi^2}{2k} \cos 2\pi(x_2 - r_0) - \frac{\pi}{4k} \sin 2\pi(x_2 - r_0), -\frac{r_0 \pi^2}{4k^2} \sin \frac{2kx_1}{r_0}, 0, 0 \right)^t, \\ \text{and } G(\psi_k^2, \psi_k^{2*}) &= \left(0, \frac{r_0 \pi^2}{4k^2} \cos \frac{2kx_1}{r_0}, -\frac{T_k^* \pi}{2} \sin 2\pi(x_2 - r_0), -\frac{q_k^* \pi}{2} \sin 2\pi(x_2 - r_0) \right)^t. \end{aligned} \quad (4.18)$$

By (4.16) and (4.17), we have

$$\langle G(\psi_k^{i_1}, \psi_k^{i_2}), \psi_k^{i_3^*} \rangle = 0, \quad (4.19)$$

for $i_1, i_2, i_3 = 1, 2$. Moreover, direct calculation shows

$$\left. \begin{aligned} \langle G(\phi_1, \phi_2), \phi_3 \rangle &= -\langle G(\phi_1, \phi_3), \phi_2 \rangle \\ \text{and} \quad \langle G(\phi_1, \psi_k^i), \psi_k^{i*} \rangle &= 0, \end{aligned} \right\} \quad (4.20)$$

for $\phi_1, \phi_2, \phi_3 \in H_1$ and $i = 1, 2$. As the centre manifold function $\Phi(x, y) = O(|x|^2 + |y|^2)$, by (4.16)–(4.20), we obtain

$$\left. \begin{aligned} \langle G(\psi), \psi_{k_c}^{1*} \rangle &= -x \langle G(\psi_{k_c}^1, \psi_{k_c}^{1*}), \Phi \rangle - y \langle G(\psi_{k_c}^2, \psi_{k_c}^{1*}), \Phi \rangle \\ &\quad + y \langle G(\Phi, \psi_{k_c}^2), \psi_{k_c}^{1*} \rangle + o(|x|^3 + |y|^3) \\ \text{and} \quad \langle G(\psi), \psi_{k_c}^{2*} \rangle &= -x \langle G(\psi_{k_c}^1, \psi_{k_c}^{2*}), \Phi \rangle - y \langle G(\psi_{k_c}^2, \psi_{k_c}^{2*}), \Phi \rangle \\ &\quad + x \langle G(\Phi, \psi_{k_c}^1), \psi_{k_c}^{2*} \rangle + o(|x|^3 + |y|^3). \end{aligned} \right\} \quad (4.21)$$

Let the centre manifold function be denoted by

$$\Phi = \sum_{\beta_{kj}^i \neq \beta_{k_c}^1} \Phi_{\beta_{kj}^i}(x, y) \psi_{kj}^i. \quad (4.22)$$

By (4.15)–(4.21), only

$$\left. \begin{aligned} \psi_{02}^1 &= (0, 0, 0, \sin 2\pi(x_2 - r_0)), \\ \psi_{02}^2 &= (0, 0, \sin 2\pi(x_2 - r_0), 0) \\ \text{and} \quad \psi_{02}^3 &= (\cos 2\pi(x_2 - r_0), 0, 0, 0) \end{aligned} \right\} \quad (4.23)$$

contribute to the third-order terms in evaluation of (4.21). Direct calculation shows that

$$\left. \begin{aligned} \langle G(\psi_{k_c}^1, \psi_{k_c}^1), \psi_{02}^1 \rangle &= -\frac{q_{k_c}}{2} \pi^2 r_0, \quad \langle G(\psi_{k_c}^1, \psi_{k_c}^2), \psi_{02}^1 \rangle = 0, \\ \langle G(\psi_{k_c}^2, \psi_{k_c}^1), \psi_{02}^1 \rangle &= 0, \quad \langle G(\psi_{k_c}^2, \psi_{k_c}^2), \psi_{02}^1 \rangle = -\frac{q_{k_c}}{2} \pi^2 r_0, \\ \langle G(\psi_{k_c}^1, \psi_{k_c}^1), \psi_{02}^2 \rangle &= -\frac{T_{k_c}}{2} \pi^2 r_0, \quad \langle G(\psi_{k_c}^1, \psi_{k_c}^2), \psi_{02}^2 \rangle = 0, \\ \langle G(\psi_{k_c}^2, \psi_{k_c}^1), \psi_{02}^2 \rangle &= 0, \quad \langle G(\psi_{k_c}^2, \psi_{k_c}^2), \psi_{02}^2 \rangle = -\frac{T_{k_c}}{2} \pi^2 r_0, \\ \langle G(\psi_{k_c}^1, \psi_{k_c}^1), \psi_{02}^3 \rangle &= 0, \quad \langle G(\psi_{k_c}^1, \psi_{k_c}^2), \psi_{02}^3 \rangle = -\frac{r_0^2 \pi^3}{2k_c} \\ \text{and} \quad \langle G(\psi_{k_c}^2, \psi_{k_c}^1), \psi_{02}^3 \rangle &= \frac{r_0^2 \pi^3}{2k_c}, \quad \langle G(\psi_{k_c}^2, \psi_{k_c}^2), \psi_{02}^3 \rangle = 0. \end{aligned} \right\} \quad (4.24)$$

We note that

$$\left. \begin{aligned} \beta_{02}^1 &= -Le \alpha_{02}^2 = -4\pi^2 Le, \\ \beta_{02}^2 &= -\alpha_{02}^2 = -4\pi^2 \\ \text{and} \quad \beta_{02}^3 &= -Pr \alpha_{02}^2 - \frac{2Pr}{r_0^2} = -4\pi^2 Pr - \frac{1}{r_0^2} Pr. \end{aligned} \right\} \quad (4.25)$$

By the approximation formula of centre manifold functions, see Section 2.4 of [23], we get

$$\left. \begin{aligned} \Phi_{\beta_{02}^1}(x, y) &= -\frac{q_k}{8\pi Le} (x^2 + y^2) + o(x^2 + y^2), \\ \Phi_{\beta_{02}^2}(x, y) &= -\frac{T_k}{8\pi} (x^2 + y^2) + o(x^2 + y^2) \\ \text{and} \quad \Phi_{\beta_{02}^3}(x, y) &= o(x^2 + y^2). \end{aligned} \right\} \quad (4.26)$$

Plugging (4.22) into (4.21), by (4.18) and (4.26), we obtain

$$\left. \begin{aligned} \langle G(\psi), \psi_{k_c}^{1*} \rangle &= -x \langle G(\psi_{k_c}^1, \psi_{k_c}^{1*}), \Phi \rangle + o(|x|^3 + |y|^3) \\ &= -\frac{\pi r_0}{16} \left(\frac{q_{k_c} q_{k_c}^*}{Le} + T_{k_c} T_{k_c}^* \right) x(x^2 + y^2) + o(|x|^3 + |y|^3) \\ \text{and} \quad \langle G(\psi), \psi_{k_c}^{2*} \rangle &= -y \langle G(\psi_{k_c}^2, \psi_{k_c}^{2*}), \Phi \rangle + o(|x|^3 + |y|^3) \\ &= -\frac{\pi r_0}{16} \left(\frac{q_{k_c} q_{k_c}^*}{Le} + T_{k_c} T_{k_c}^* \right) y(x^2 + y^2) + o(|x|^3 + |y|^3). \end{aligned} \right\} \quad (4.27)$$

By (4.10), (4.13) and (4.27), we evaluate (4.7) at $R = R_c$ and obtain the reduction equation

$$\left. \begin{aligned} \frac{dx}{dt} &= -\frac{b}{8} x(x^2 + y^2) + o(|x|^3 + |y|^3) \\ \text{and} \quad \frac{dy}{dt} &= -\frac{b}{8} y(x^2 + y^2) + o(|x|^3 + |y|^3), \end{aligned} \right\} \quad (4.28)$$

where b is as defined as

$$b = \frac{Le^3 \alpha_{k_c}^2 R_c + (((1 + Le^2) \alpha_1 \alpha_T h^2) / \alpha_q \kappa_T + \alpha_1 \alpha_T \alpha_{k_c}^2 Le^3 h^2 / 24 \alpha_q \kappa_T + \alpha_{k_c}^2) \tilde{R}}{(Le^3 \alpha_{k_c}^6 / Pr)(1 + r_0^2 \pi^2 / k_c^2) + Le^3 \alpha_{k_c}^2 R + Le((1 + Le) \alpha_1 \alpha_T h^2 / \alpha_q \kappa_T + \alpha_1 \alpha_T \alpha_{k_c}^2 Le^2 h^2 / 24 \alpha_q \kappa_T + \alpha_{k_c}^2) \tilde{R}}, \quad (4.29)$$

and $\alpha_{k_c}^2$ and R_c are given by remark 3.5. It is obvious that $b > 0$. As $b > 0$, due to the reduction equation (4.28), assertion (2) follows from Theorem 2.2.5 of [24]. Assertions (1) and (3) follow from Theorem 3.1 and Theorem 2.4.17 of [24]. This completes the proof of theorem 4.1. ■

5. Convection scales

Under the same setting as (2.17)–(2.19), including the fluid frictions, we consider the following non-dimensional equation:

$$\left. \begin{aligned} \frac{\partial u_1}{\partial t} &= Pr \left(\Delta u_1 + \frac{2}{r_0} \frac{\partial u_2}{\partial x_1} - \frac{1}{r_0^2} u_1 - \delta_0 u_1 - \frac{\partial p}{\partial x_1} \right) \\ &\quad - \omega u_2 - (u \cdot \nabla) u_1 - \frac{1}{r_0} u_1 u_2, \\ \frac{\partial u_2}{\partial t} &= Pr \left(\Delta u_2 - \frac{2}{r_0} \frac{\partial u_1}{\partial x_1} - \frac{2}{r_0^2} u_2 + RT + \tilde{R} q - \delta_1 u_2 - \frac{\partial p}{\partial x_2} \right) \\ &\quad + \omega u_1 - (u \cdot \nabla) u_2 + \frac{1}{r_0} u_1^2, \\ \frac{\partial T}{\partial t} &= \Delta T + \gamma u_2 + \alpha q - (u \cdot \nabla) T, \\ \frac{\partial q}{\partial t} &= Le \Delta q + u_2 - (u \cdot \nabla) q \\ \text{and} \quad \operatorname{div} u &= 0, \end{aligned} \right\} \quad (5.1)$$

where

$$\left. \begin{aligned} \gamma &= 1 + \frac{\alpha}{24}, \\ \delta_i &= \frac{C_i h^4}{\nu}, \quad i = 0, 1, \\ C_0 &= 3.78 \times 10^{-1} \text{ m}^{-2} \text{ s}^{-1} \\ \text{and} \quad C_1 &= 6.7 \times 10^4 \text{ m}^{-2} \text{ s}^{-1}. \end{aligned} \right\} \quad (5.2)$$

The domain is $M = [0, 2\pi r_0] \times (r_0, r_0 + 1)$, and the boundary conditions are given by

$$\left. \begin{aligned} (u, T, q)(x_1 + 2\pi r_0, x_2) &= (u, T, q)(x_1, x_2) \\ \text{and} \quad u_2 = 0, \quad \frac{\partial u_1}{\partial x_2} &= 0, \quad T = 0, \quad q = 0, \quad \text{at } x_2 = r_0, r_0 + 1. \end{aligned} \right\} \quad (5.3)$$

The corresponding eigenvalue problem reads

$$\left. \begin{aligned} Pr \left(\Delta u_1 + \frac{2}{r_0} \frac{\partial u_2}{\partial x_1} - \frac{1}{r_0^2} u_1 - \delta_0 u_1 - \frac{\partial p}{\partial x_1} \right) - \omega u_2 &= \beta u_1, \\ Pr \left(\Delta u_2 - \frac{2}{r_0} \frac{\partial u_1}{\partial x_1} - \frac{2}{r_0^2} u_2 + RT + \tilde{R}q - \delta_1 u_2 - \frac{\partial p}{\partial x_2} \right) + \omega u_1 &= \beta u_2, \\ \Delta T + \gamma u_2 + \alpha q &= \beta T, \\ Le \Delta q + u_2 &= \beta q \\ \text{and} \quad \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} &= 0. \end{aligned} \right\} \quad (5.4)$$

Using the same analysis as in §3a, we obtain the following formula of the critical thermal Rayleigh number:

$$\begin{aligned} R_c &= \min_{k, j \geq 1} \left[- \left(\frac{1}{Le} + \left(1 + \frac{Le \alpha_{kj}^2}{24} \right) \frac{\alpha_1 \alpha_T h^2}{Le \alpha_q \alpha_{kj}^2 \kappa_T} \right) \tilde{R} \right. \\ &\quad \left. + \alpha_{kj}^4 \frac{r_0^2}{k^2} \left(\alpha_{kj}^2 + \frac{1}{r_0^2} \right) + \frac{\alpha_{kj}^2}{r_0^2} + \delta_1 \alpha_{kj}^2 + \frac{\delta_0 \pi^2 r_0^2 \alpha_{kj}^2}{k^2} \right] \\ &= \min_{k \geq 1} \left[- \left(\frac{1}{Le} + \left(1 + \frac{Le \alpha_{k1}^2}{24} \right) \frac{\alpha_1 \alpha_T h^2}{Le \alpha_q \alpha_{k1}^2 \kappa_T} \right) \tilde{R} \right. \\ &\quad \left. + \alpha_{k1}^4 \frac{r_0^2}{k^2} \left(\alpha_{k1}^2 + \frac{1}{r_0^2} \right) + \frac{\alpha_{k1}^2}{r_0^2} + \delta_1 \alpha_{k1}^2 + \frac{\delta_0 \pi^2 r_0^2 \alpha_{k1}^2}{k^2} \right], \end{aligned} \quad (5.5)$$

where

$$\alpha_{k1}^2 = \pi^2 + \frac{k^2}{r_0^2}.$$

Let, $y = k^2/r_0^2$ and

$$\begin{aligned} g(y) &= - \left(\frac{1}{Le} + \left(1 + \frac{Le(y + \pi^2)}{24} \right) \frac{\alpha_1 \alpha_T h^2}{Le \alpha_q \kappa_T (y + \pi^2)} \right) \tilde{R} + \frac{1}{y} (y + \pi^2)^2 \left(y + \pi^2 + \frac{1}{r_0^2} \right) \\ &\quad + \frac{1}{r_0^2} (y + \pi^2) + \delta_1 (y + \pi^2) + \frac{\delta_0 \pi^2 (y + \pi^2)}{y}. \end{aligned}$$

Taking the derivative of $g(y)$, we get

$$g'(y) = 2y + \left(3\pi^2 + \frac{2}{r_0^2} + \delta_1 \right) - \frac{1}{y^2} \left(\pi^6 + \frac{\pi^4}{r_0^2} + \delta_0 \pi^4 \right) + \frac{\alpha_1 \alpha_T h^2 \tilde{R}}{Le \alpha_q \kappa_T (y + \pi^2)^2}.$$

By (5.2), we have

$$\delta_1 = 2.76 \times 10^{20} \quad \text{and} \quad \delta_0 = 1.55 \times 10^{15}. \quad (5.6)$$

Hence,

$$\delta_1 \gg \delta_0 \gg 1. \quad (5.7)$$

Under this condition, the critical value of $g(y)$ is approximated by

$$y_c \simeq \left(\frac{\delta_0}{\delta_1} \right) \pi^2. \quad (5.8)$$

Therefore, the critical thermal Rayleigh number is approximated by

$$R_c \simeq \delta_1 \pi^2 = 2.76 \times 10^{21}. \quad (5.9)$$

Next, as in [18,24], we adopt

$$\left. \begin{aligned} \nu &= 1.6 \times 10^{-5} \text{ m}^2 \text{ s}^{-1}, \quad \kappa_T = 2.25 \times 10^{-5} \text{ m}^2 \text{ s}^{-1}, \\ \text{and} \quad \alpha_T &= 3.3 \times 10^{-3} \text{ per } ^\circ\text{C}, \quad Pr = 0.71. \end{aligned} \right\} \quad (5.10)$$

We note also that the height of the troposphere is

$$h = 8 \times 10^3 \text{ m}.$$

Hence, we get

$$T_0 - T_1 = \Delta T_c = \frac{\kappa_T \nu}{\alpha_T g h^3} R_c = 60^\circ\text{C}. \quad (5.11)$$

Here, we note that the above approximations agree with that of the model of tropical atmospheric circulations without humidity. However, as we can see from (5.5), the coefficient of \tilde{R} is negative. This implies that the humidity factor lowers the critical thermal Rayleigh number a little bit.

Next, as the non-dimensional radius of the Earth is $r_0 = 6400000/h = 800$, we derive from

$$\frac{k^2}{r_0^2} = \left(\frac{\delta_0}{\delta_1} \right)^{1/2} \pi^2 \quad (5.12)$$

that the wavenumber k_c and the convection length-scale L_c as

$$k_c \simeq 6 \quad \text{and} \quad L = \frac{(6400 \times \pi)}{6} = 3350 \text{ km}.$$

This is consistent with the large-scale atmospheric circulation over the tropics.

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Appendix A. Recapitulation of dynamic transition theory

The main results of this article are based on the dynamic transition theory developed recently by two of the authors [23,24]. Hereafter, we briefly recapitulate the basic ideas of the theory and refer the interested readers to these references for more details.

First, dissipative systems are governed by differential equations—both ordinary and partial—which can be written in the following unified abstract form:

$$\frac{du}{dt} = L_\lambda u + G(u, \lambda), \quad u(0) = u_0, \quad (\text{A } 1)$$

where $u : [0, \infty) \rightarrow X$ is the unknown function, $\lambda \in \mathbb{R}^1$ is the system control parameter and X is a Banach space. We shall always consider u the deviation of the unknown function from some equilibrium state \bar{u} . Hence, $L_\lambda : X_1 \rightarrow X$ is a linear operator, and $G : X_1 \times \mathbb{R}^1 \rightarrow X$ is a nonlinear operator. For example, for the classical incompressible Navier–Stokes equations, u represents the velocity function, and for the time-dependent Ginzburg–Landau equations of superconductivity, we have $u = (\psi, A)$, where ψ is a complex-valued wave function, and A is the magnetic potential.

Second, the dynamic transition of a given dissipative system is clearly associated with the linear eigenvalue problem for system (A 1). The underlying physical concept is the PES, leading to

precise information on linear unstable modes. Let, $\{\beta_j(\lambda) \in \mathbb{C} \mid j \in \mathbb{N}\}$ be the eigenvalues (counting multiplicity) of L_λ and assume that

$$\operatorname{Re} \beta_i(\lambda) \begin{cases} < 0 & \text{if } \lambda < \lambda_0, \\ = 0 & \text{if } \lambda = \lambda_0, \\ > 0 & \text{if } \lambda > \lambda_0 \end{cases} \quad \forall 1 \leq i \leq m, \quad (\text{A } 2)$$

and

$$\operatorname{Re} \beta_j(\lambda_0) < 0 \quad \forall j \geq m + 1. \quad (\text{A } 3)$$

Third, with PES, the dynamic transition is fully dictated then by the nonlinear interactions of the system. One important component of the dynamic transition theory is to establish a general principle, which classifies all dynamic transitions of a dissipative system into three categories, continuous, catastrophic and random, which are also called Type I, Type II and Type III. A continuous transition says that as the control parameter crosses the critical threshold, the transition states stay in a close neighbourhood to the basic state. A catastrophic transition corresponds to the case in which the system undergoes a more drastic change as the control parameter crosses the critical threshold. A random transition corresponds to the case in which a neighbourhood (fluctuations) of the basic state can be divided into two regions such that fluctuations in one of them lead to continuous transitions and those in the other lead to catastrophic transitions.

Fourth, in the dynamic transition theory, the complete set of transition states is represented by local attractors. The identification and classification of these local attractors are important part of the theory. One crucial technical component is the approximation of the centre manifold of the underlying system, corresponding to the m unstable modes as described in the PES.

In fact, using the centre manifold reduction, we know that the type of transitions for (A 1) at $(0, \lambda_0)$ is completely dictated by its reduction equation near $\lambda = \lambda_0$

$$\frac{dx}{dt} = J_\lambda x + g(x, \lambda) \quad \text{for } x \in \mathbb{R}^m, \quad (\text{A } 4)$$

where $g(x, \lambda) = (g_1(x, \lambda), \dots, g_m(x, \lambda))$ and

$$g_j(x, \lambda) = \left\langle G \left(\sum_{i=1}^m x_i e_i + \Phi(x, \lambda), \lambda \right), e_j^* \right\rangle \quad \forall 1 \leq j \leq m. \quad (\text{A } 5)$$

Here, e_j and e_j^* ($1 \leq j \leq m$) are the eigenvectors of L_λ and L_λ^* , respectively, corresponding to the eigenvalues $\beta_j(\lambda)$, J_λ is the $m \times m$ order Jordan matrix corresponding to the eigenvalues, and $\Phi(x, \lambda)$ is the centre manifold function of (A 1) near λ_0 .

The centre manifold function Φ is implicitly defined and is oftentimes hard to compute. A systematic approach is developed in [23,24] to derive approximations of Φ , which provide complete information on the dynamic transition of (A 1). Suppose the nonlinear operator G to be of the form

$$G(u, \lambda) = G_k(u, \lambda) + o(\|u\|^k), \quad \text{as } u \rightarrow 0 \text{ in } X_\mu. \quad (\text{A } 6)$$

for some integer $k \geq 2$, where G_k is a k -multilinear operator

$$G_k(u, \lambda) = G_k(u, \dots, u, \lambda) : X_1 \times \dots \times X_1 \longrightarrow X.$$

By Theorem A.1.1 in [24], the centre manifold function $\Phi(x, \lambda)$ can be expressed as

$$\Phi(x, \lambda) = \int_{-\infty}^0 e^{-\tau \mathcal{L}_\lambda} \rho_\varepsilon P_2 G_k(e^{\tau J_\lambda} x, \lambda) d\tau + o(\|x\|^k), \quad (\text{A } 7)$$

where J_λ and \mathcal{L}_λ are restrictions of L_λ to the linear invariant spaces generated by the first modes and the rest modes, respectively, $G_k(u, \lambda)$ is the lowest order k -multiple linear operator, and $x = \sum_{i=1}^m x_i e_i$. In particular, we have the following assertions:

(1) If J_λ is diagonal near $\lambda = \lambda_0$, then (A 7) can be written as

$$-\mathcal{L}_\lambda \Phi(x, \lambda) = P_2 G_k(x, \lambda) + o(\|x\|^k) + O(|\beta| \|x\|^k). \quad (\text{A } 8)$$

(2) Let $m = 2$ and $\beta_1(\lambda) = \overline{\beta_2(\lambda)} = \alpha(\lambda) + i\rho(\lambda)$ with $\rho(\lambda_0) \neq 0$. If $G_k(u, \lambda) = G_2(u, \lambda)$ is bilinear, then $\Phi(x, \lambda)$ can be expressed as

$$\begin{aligned} [(-\mathcal{L}_\lambda)^2 + 4\rho^2(\lambda)](-\mathcal{L}_\lambda)\Phi(x, \lambda) = & [(-\mathcal{L}_\lambda)^2 + 4\rho^2(\lambda)]P_2 G_2(x, \lambda) \\ & - 2\rho^2(\lambda)P_2 G_2(x, \lambda) + 2\rho^2 P_2 G_2(x_1 e_2 - x_2 e_1) \\ & + \rho(-\mathcal{L}_\lambda)P_2 [G_2(x_1 e_1 + x_2 e_2, x_2 e_1 - x_1 e_2) \\ & + G_2(x_2 e_1 - x_1 e_2, x_1 e_1 + x_2 e_2)] + o(k), \end{aligned} \quad (\text{A } 9)$$

where we have used $o(k) = o(\|x\|^k) + O(|\operatorname{Re} \beta(\lambda)| \|x\|^k)$.

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