The concepts of convexity and entropy play a crucial role in the mathematical theory of hyperbolic systems of conservation laws. We show that they also play an important role in the mathematical analysis of convection theory, through the mathematical concept of rearrangement.

1. Introduction

Rearrangement theory is about reorganizing a given function (or map) in some specific order, in particular as an increasing function (or a map with convex potential). This is somewhat similar to the convection of a fluid, where fluid parcels are continuously reorganized in a stabler way (heavy fluid at bottom and light fluid at top) under the action of the buoyancy force. Convection is one of the most important phenomenon in nature (atmosphere, ocean, continent drift, volcanism, earth quakes, etc) and, of course, plays a crucial role in our daily life (heating, cooking, etc). In this paper, we establish several connections, somewhat unexpected, between convection, rearrangements, and the concepts of convexity and entropy. Both these concepts play a crucial role in one of the most important fields in nonlinear PDE theory and applied analysis, namely the theory of hyperbolic systems of conservation laws (as described, for instance, in Dafermos’ book [1]). They are loosely related to shock waves in compressible gas dynamics and, ultimately, to the underlying Boltzmann–Maxwell kinetic theory of gases. It is rather surprising that both convexity and entropy could play any significant role in convection theory, which is mostly related to the motion of incompressible fluids with no relation to shock waves. To establish such connections, we start our discussion with a very simple and crude model of one-dimensional (‘column’) convection based on elementary rearrangement theory. Rather quickly, we already see that this model can be described
as a one-dimensional, hyperbolic scalar conservation law with monotonic initial conditions. Here comes the first occurrence of ‘entropy solutions’ in our discussion. Next, we investigate a multi-dimensional generalization of the model, which is natural from a mathematical (rather than physical) point of view. This generalization is based on the concept of rearrangement of maps as maps with convex potential [2,3], crucially related to the now widespread theory of optimal transportation (as described in Villani’s books [4,5]). It turns out that our multi-dimensional model can physically be interpreted as a model of (generalized) convection with small source terms acting on long time scales (and, therefore, can be seen—without any pretention from our side—as a toy model for ‘global change’ modelling). The model is obtained as a scaling limit of the Navier–Stokes equation with (generalized) buoyancy forces, that we call ‘Hydrostatic Boussinesq’ (HB) equations. For the HB equations, we get a concept of ‘entropy solutions’ involving a convexity condition, which essentially coincides with the Cullen–Purser convexity condition for semigeostrophic flows [6]. Singularities may occur in finite time, as strong convexity breaks down, which is, of course, reminiscent of shock formation in compressible gas dynamics. Mathematically speaking, we provide a global existence theory for ‘entropy solutions’, under the convexity condition, but do not get uniqueness results at this stage. In addition, before singularities occur, we can show that the solution of the HB equations are indeed the correct limit of the solutions of the complete Navier–Stokes–Boussinesq (NSB) system. To achieve this later result, we use a ‘relative entropy’ method involving the (strong) convexity condition in a crucial role.

2. A one-dimensional model of convection related to hyperbolic scalar conservation laws

(a) The concept of rearrangement in increasing order

Let us recall a well-known result of analysis (see [7] for example). Any $L^2$ real-valued function $x \in [0, 1] \rightarrow z(x)$ admits a unique rearrangement in increasing order, i.e. a unique non-decreasing $L^2$ function $z^♯$ such that

$$
\int_{[0,1]} f(z^♯(x)) \, dx = \int_{[0,1]} f(z(x)) \, dx \tag{2.1}
$$

for all continuous function $f$ with at most quadratic growth.

Note that in the discrete case when

$$
z(x) = Z_j, \quad \frac{j}{N} < x < \frac{(j+1)}{N}, \quad j = 0, \ldots, N - 1 \tag{2.2}
$$

then $z^♯(x) = Z^♯_j$ where $(Z^♯_1, \ldots, Z^♯_N)$ is just $(Z_1, \ldots, Z_N)$ sorted in increasing order.

An example of (discrete) rearrangement (with 500 grid points) is shown in figure 1.

(b) A crude model of column convection

A crude model of convection (somewhat reminiscent of ‘single column models’ in meteorological sciences [8]) can be described as follows. Let us consider a vertical column with coordinate $x \in [0, 1]$. We assume the column to be heated periodically in time: at every time step $h > 0$, a heat source acts along the column at each elevation $x$ and increases the temperature by $G = G(x)$ (which could also be time-dependent). We denote by $y_n(x)$ the temperature at elevation $x$ and time $nh$. Although this model will be analysed in the limit $h \rightarrow 0$, we assume $h$ to be sufficiently long so that the temperature profile rearranges itself in the stabelst possible manner, namely in increasing order at the end of each time step. Then, the convection mechanism is entirely described by

$$
y_{n+1} = (\tilde{y}_{n+1})^♯, \quad \tilde{y}_{n+1}(x) = y_n(x) + hG(x),
$$

where $^♯$ denotes the rearrangement operator described in §2a. Note that, if $G$ is piecewise constant on subintervals of constant elevation, the model reduces to a fully discrete scheme, without
further modification. Therefore, numerical experiments can be easily performed, just by using a fast sorting algorithm at each time step.

Let us now look at numerical simulations. We use 
\[ G(x) = 1 + \exp(-25(x - 0.2)^2) - \exp(-20(x - 0.4)^2) \]
(with a positive source near \( x = 0.2 \) and a negative one near \( x = 0.4 \)). More precisely, we use the piecewise constant approximation of \( G \) obtained on 500 subintervals of equal length. In our first experiment, we set \( h = 0.1 \) and we plot the temperature profiles (before and after stabilization) at every two time steps (i.e. at \( t = 2nh - 0 \) and \( t = 2nh + 0 \), for \( n = 0, 1, 2, \ldots \)). The result is shown in figure 2. In the second experiment, we use a much finer time step \( h = 0.005 \) and we plot the corresponding temperature profiles at every 40 time steps. We also plot the time-dependent zone where the rearrangement operator acts. This can be seen as a figure of the mixing zone where molecules rearrange themselves along the column in order to make the temperature profile as stable as possible. These results are shown in figures 3 and 4.
Figure 3. Temperature profiles with a fine time step \( h = 0.005 \).

Figure 4. Space–time mixing zone.

### (c) Remark: a possible social science interpretation

It is worth, in our opinion, providing a ‘social science’ interpretation of our crude model of column convection. The model is as follows: \( N \) agents (factories, researchers, universities, etc.) are in competition. We denote by \( x_n(i) \) the cumulated production of agent \( i = 1, \ldots, N \) at time \( nh \) and by \( \sigma_n(i) \) the (reverse) rank of agent \( i \) at time \( nh \). Then, the evolution is described by

\[
x_{n+1}(i) = x_n(i) + hG \left( \frac{\sigma_n(i)}{N} \right),
\]

where \( G \) is a given function. Observe that the corresponding sorted sequence \( y_n = x_n^\# \) satisfies:

\[
y_{n+1} = (y_n + hG)^2,
\]

which is just the space-discrete version of our rearrangement scheme.

In this model, the rate of production between two different times depends only on the ranking. For example \( G(x) = 1 - x \), means that the top people slow down their production while the bottom people catch up as fast as possible. The choice \( G(x) = 1 - \cos(3\pi x) \) seems more realistic: bottom people are discouraged while top people get even more competitive \( G(0) = 0, G(\frac{1}{3}) = 2, G(\frac{2}{3}) = 0, G(1) = 2 \). We believe that this very simple model could be useful (if not already known!) in economic sciences. (More precisely, it would be interesting to ‘identify’ function \( G \) from empirical observations.)
(d) Convergence analysis and scalar conservation laws

Using the classical theory of maximal monotone operators [9], it is fairly easy to prove

**Theorem 2.1.** As $h \to 0$, the time-discrete scheme has a unique limit $y = y(t, x)$, monotonically increasing in $x$, characterized as the unique solution in $C^0(\mathbb{R}_+, L^2(D, \mathbb{R}^d))$ of the subdifferential inclusion

$$G(x) \in \partial y + \partial C[y], \quad y(t = 0, \cdot) = y_0,$$

(2.3)

where $C[y] = 0$ or $+\infty$, according to whether or not $y$ is a non-decreasing function of $x$.

More interestingly, for our purposes, there is a strong link with the theory of ‘entropy solutions’ for ‘scalar conservation laws’ [1]. More precisely, the cumulative function $u(t, s) = \int_0^1 1\{y(t, x) < s\} \, dx$, which is the ‘pseudo-inverse’ function of $y$, is an ‘entropy’ solution to the scalar conservation law

$$\partial_t u + \partial_s (g(u)) = 0, \quad g(v) = \int_0^v G(w) \, dw.$$  (2.4)

This has been proved a long time ago [10]: indeed, the scheme we have described is nothing but the ‘transport-collapse’ method for solving (2.4).

**Remark 2.2.** This double convergence result suggests the possibility of solving multi-dimensional scalar conservation laws in the framework of maximal monotone operators in $L^2$. This is, indeed, possible and was done in [11].

(e) Shock formation

Scalar conservation laws such as $\partial_t u + \partial_s (g(u)) = 0$ are known to produce in finite time solutions $s \to u(t, s)$ with discontinuities, known as ‘shock waves’, yielding the famous concept of ‘entropy’ solutions, inherited from gas dynamics [1]. For the temperature field $x \to y(t, x)$, this means the formation of a plateau, which corresponds to a zone where the temperature field is homogenized. In the canonical example $G(x) = G(x) = 1 - x$, corresponding to the famous ‘inviscid’ Burgers equation $\partial_t u + \partial_x (u - u^2/2) = 0$, it can be shown that, for all initial conditions, a single plateau forms in finite time, which corresponds to a perfectly homogenized temperature. For functions like $G(x) = 1 - \cos(3\pi x)$, the long-time behaviour will be described by a middle plateau surrounded by two tails, one cold at bottom and one hot at top. (An obvious interpretation in terms of social sciences being the formation of social classes.)

3. Multi-dimensional generalization of the rearrangement scheme

(a) Multi-dimensional rearrangement and optimal transportation

The rearrangement of scalar functions in increasing order, mentioned at the very beginning of the paper, is a classical tool in Analysis. This concept admits a natural vectorial and multi-dimensional generalization through the concept (now well established) of rearrangement as maps with convex potentials. More precisely:

**Theorem 3.1.** Given a bounded domain $D \subset \mathbb{R}^d$ and an $L^2$ map $x \in D \to z(x) \in \mathbb{R}^d$, there is a unique rearrangement with convex potential $z^2(x) = \nabla p(x)$, where $p(x)$ is lsc convex in $x \in \mathbb{R}^d$, a.e. differentiable on $D$, such that

$$\int_D f(\nabla p(x)) \, dx = \int_D f(z(x)) \, dx,$$

(3.1)

for all continuous function $f$ such that $|f(x)| \leq \text{cst}(1 + |x|^2)$

This result, obtained in [2,3,13], is a typical result in optimal transport theory [4] and has many possible versions, generalizations and applications.
(b) Multi-dimensional generalization of the rearrangement scheme

The concept of rearrangement of maps as maps with convex potential immediately suggests the following multi-dimensional generalization of our elementary rearrangement scheme. We consider

— a smooth bounded domain \( x \in D \subset \mathbb{R}^d \),
— a vector-valued field: \( y(t, x) \in \mathbb{R}^d \) (generalized temperature), and
— a vector-valued source term: \( G(x) \in \mathbb{R}^d \) with bounded derivatives.

Then we introduce a time discrete scheme, with time step \( h > 0 \), where we approximate \( y(t = nh, x) \) by \( y_n(x) \), for \( n = 0, 1, 2, \ldots \), as follows:

\[
\begin{align*}
\bar{y}_{n+1}(x) &= y_n(x) + hG(x), \quad y_{n+1} = \bar{y}_{n+1} = \nabla p_{n+1},
\end{align*}
\]  

(3.2)

where the second step amounts to rearranging a map as a map with convex potential.

(c) Interpretation of the multi-dimensional rearrangement scheme

The multi-dimensional scheme we have just defined is a priori just an artificial generalization. Can we say more about it? We may discuss several natural questions, such as:

— Does the scheme correspond to a realistic, genuinely multi-dimensional, model of convection?
— Does it provide an interesting multi-dimensional description of shock waves and ‘entropy solutions’?
— Has it an interesting interpretation in terms of social sciences?

Subsequently, we are going to address the first issue: it turns out that, indeed, the scheme can be interpreted as a singular limit of an NSB model with (generalized) buoyancy forces. This is what we are now going to explain in the second part of the paper.

4. Hydrostatic limit of the Navier–Stokes–Boussinesq equations

(a) A multi-dimensional model of convection

Let \( D \) be a smooth bounded domain \( D \subset \mathbb{R}^3 \) in which moves an incompressible fluid of velocity \( v(t, x) \) at \( x \in D, t \geq 0 \), subjected to the NSB equations

\[
(\partial_t v + v \cdot \nabla)v - \alpha \nabla^2 v + \nabla p = y, \quad (\partial_t + v \cdot \nabla)y = \epsilon G(\epsilon t, x)
\]  

(4.1)

with \( \nabla \cdot v = 0 \) and \( v = 0 \) along \( \partial D \).

The field \( y = y(t, x) \in \mathbb{R}^3 \) is a ‘generalized buoyancy’, vector-valued force, with a small, slowly evolving source term, where \( G \) is a given smooth function with bounded derivatives. (In usual coordinates, the buoyancy force acts only along the vertical axis \( x_3 \), so that \( G_1 = G_2 = 0 \). ‘Generalized buoyancy forces’ can be obtained, on top of the usual buoyancy force, either by considering the Coriolis force (in which case one of the horizontal coordinates must be neglected—see below) or by applying a constant electric field along one of the horizontal coordinates in the case of an electrically charged fluid.) We can see these equations as a (very crude!) model for a global climate change: we look for the long-time impact of a small, slowly evolving, source term of amplitude \( \epsilon \) on long time scales of order \( \epsilon^{-1} \).
By substituting \((t, v, p, y)\) for \((\epsilon t, \epsilon v, p, y)\) in the NSB equations (4.1), we get the following rescaled Navier–Stokes–Boussinesq (RNSB) equations:

\[
y = \nabla p + \epsilon^2 (\partial_t v + (v \cdot \nabla)v) - \alpha \epsilon \nabla^2 v, \quad \nabla \cdot v = 0, \quad \partial_t y + (v \cdot \nabla)y = G(t, x). \tag{4.2}
\]

We call ‘HB’ equations, the formal limit obtained for \(\epsilon = 0\)

\[
y = \nabla p, \quad \nabla \cdot v = 0, \quad \partial_t y + (v \cdot \nabla)y = G(t, x). \tag{4.3}
\]

Remark 4.1. In the more precise convection model we considered with Mike Cullen [14], there is no \(x_2\) dependence and \(G_1 = 0\). Then the force field \(y\) is vector valued and combines both Coriolis (in the \(x_1\) direction) and buoyancy (in the \(x_3\) direction) effects. The \(\epsilon \to 0\) limit is, then, related to Brian Hoskins’ ‘x-z’ semigeostrophic equations [6,15].

Remark 4.2. From the PDE viewpoint, global existence of weak solutions in three dimensions follows from Hou & Li [17] and Chae [18]. (See also a more refined work by Danchin & Paicu [19].)

Remark 4.3. For any suitable test function \(f\), we have INDEPENDENTLY of \(\epsilon, v\) the following key property:

\[
\frac{d}{dt} \int_D f(y(t, x)) \, dx = \int_D (\nabla f)(y(t, x)) \cdot G(t, x) \, dx \tag{4.4}
\]

This is valid even for the Leray weak solutions (thanks to DiPerna–Lions’ theory on ODEs [20,21]).

Remark 4.4. When both the source term and the initial force are gradients and the fluid initially is at rest

\[
G = G(x) = \nabla g(x), \quad y(0, x) = \nabla p_0(x), \quad v(0, x) = 0, \tag{4.5}
\]

then the rescaled NSB system (4.2) has a trivial but interesting ‘convection-free’ solution, independently of \(\epsilon\), namely

\[
v(t, x) = 0, \quad y(t, x) = \nabla p(t, x), \quad p(t, x) = p_0(x) + tg(x). \tag{4.6}
\]

Of course, these solutions are also trivial solutions to HB system (4.3).

(b) A natural convexity condition for the hydrostatic Boussinesq system

HB system (4.3), formally obtained by setting \(\epsilon\) to zero in the RNSB equations (4.2) looks strange because there is no evolution equation for \(v\). However, we have a constraint for \(y\), namely to be a gradient, since \(y = \nabla p\). Thus, we can recover \(v\) as a kind of Lagrange multiplier of this constraint. Indeed, note first that \((v \cdot \nabla)y = (D^2_{xy} p \cdot v)\) and \(v = \nabla \times A\), for some divergence-free vector potential \(A = A(t, x) \in \mathbb{R}^3\), when \(d = 3\). Then, taking the curl of the evolution equation in HB system (4.3), we get

\[
\nabla \times (D^2_{xy} p(t, x) \cdot \nabla \times A) = \nabla \times G. \tag{4.7}
\]

At each fixed time \(t\), this is a linear ‘magnetostatic’ system in \(A\), which is elliptic whenever \(p\) is convex in the strong sense

\[
c \text{Id} < D^2_{xy} p(t, x) < c^{-1} \text{Id}, \quad \forall x, \tag{4.8}
\]

for some constant \(c > 0\). This strongly suggests that the HB system is well posed, under this strong convexity assumption (4.8), which, presumably, is sustainable, at least on short time intervals. This is indeed correct and was proved by Loeper (for a specific choice of \(G\), but his method goes through the general case of a smooth function \(G\) with bounded derivatives), using a Monge–Ampère reformulation of the system [22]. The proof has been obtained by Loeper only in the case of a periodic domain, such as \(D = (\mathbb{R}/\mathbb{Z})^3\). This periodic setting requires a little bit of care: the
pressure $p(t, x)$ should be understood as the sum of $|x|^2/2$ and a 1-periodic function $p'(t, x)$, the strong convexity condition meaning

$$c \text{Id} < \text{Id} + D^2_p(t, x) < c^{-1} \text{Id}, \quad \forall x,$$

(4.9)

for some constant $c > 0$. Accordingly, $y(t, x) - x = \nabla p'(t, x)$ is also a 1-periodic, vector-valued function, just as $v(t, x)$. Note that condition (4.9) implies that the Legendre–Fenchel transform of $p$, defined by

$$p^*(t, y) = \sup_{x \in \mathbb{R}^d} x \cdot y - p(t, x),$$

(4.10)

also satisfies

$$c \text{Id} < D^2_p^*(t, y) < c^{-1} \text{Id}, \quad \forall y.$$

(4.11)

As a consequence, both $x \to \nabla p(t, x)$ and $y \to \nabla p^*(t, y)$ define global orientation-preserving diffeomorphisms of $\mathbb{R}^3$.

(c) Rigorous derivation of the hydrostatic Boussinesq model under strong convexity condition

The strong condition (4.8) is sufficient to get a rigorous derivation of HB equations (4.3) from RNSB equations (4.2) as $\epsilon$ goes to zero, at least in the case of a periodic domain.

**Theorem 4.5.** Let $D = \mathbb{R}^3/\mathbb{Z}^3$. Assume $G$ to be smooth with bounded derivatives up to second order. Let $(y^\epsilon, v^\epsilon, p^\epsilon)$ be a Leray-type solution to RNSB equations (4.2). Let $(y = \nabla p, v)$ be a smooth solution to HB equations (4.3) on a given finite time interval $[0, T]$. We assume that the strong convexity conditions (4.8) and (4.9) are satisfied up to time $T$.

Then, the $L^2$ distance between $y^\epsilon$ and $y$ stays uniformly of order $\sqrt{\epsilon}$ as $\epsilon$ goes to zero, uniformly in $t \in [0, T]$, provided it does at $t = 0$ and the initial velocity $v^\epsilon(t = 0, x)$ stays uniformly bounded in $L^2$.

This result was claimed in our short note [14] for general domains and no-slip boundary conditions, but there is a flaw in the proof, where boundary terms have not been treated correctly. (We thank Helena Lopes-Nussenzveig for pointing out the problem to us.) Of course, this can be easily fixed, as usual, if Navier slip conditions substitute for the classical no-slip condition along the boundary, or, alternatively, as we do here, in the case of periodic boundary conditions. For the sake of clarity, a complete corrected proof is delivered in appendix A of this paper. At the moment, let us just tell the key idea of the proof. A natural idea, to compare $y^\epsilon$ and $y$, would be to estimate $L^2$ (or more generally Sobolev) norms and try to get Gronwall-type differential inequalities for them. This method completely fails, owing to the presence of an irreducible term of size $\epsilon^{-1}$.

The right idea is use a ‘relative entropy’ method, loosely borrowed from the theory of ‘entropy solutions’ for hyperbolic systems of conservation laws, and in particular from Dafermos’ ‘weak–strong’ uniqueness method [1]. (As a matter of fact, the version we use here is quite similar to the one used by the author for the hydrostatic limit of the two-dimensional Euler equations in a thin domain [13].) We introduce a properly chosen Lyapunov-type functional (also called ‘relative entropy’ or ‘modulated energy’, or ‘Bregman functional’, by different authors) in order to compare solutions of the HB limit model to those of the original RNSB equations. We set

$$\int_D \left\{K(t, y^\epsilon(t, x), y(t, x)) + \frac{\epsilon^2}{2} |v^\epsilon - v|^2\right\} \text{d}x,$$

(4.12)

where

$$K(t, y', y) = p^*(t, y') - p^*(t, y) - \nabla p^*(t, y) \cdot (y' - y) \sim |y - y'|^2,$$

(4.13)

where $p^*$ is the Legendre–Fenchel transform of $p$ defined by (4.10). Under the strong convexity assumption (4.8), this functional controls the squared $L^2$ distance between $y^\epsilon$ and $y$. Then, we are able to show that, as long as the strong convexity assumption holds true for the limit $y$,
the functional is well controlled by its initial value, which shows the $L^2$ convergence of $y^\epsilon$ towards $y$ as $\epsilon$ goes to zero. Note how this result combines convection, convexity and entropy (through the concepts of ‘relative entropy’ and ‘entropy solutions’).

**d) Breakdown of convexity and a concept of ‘entropy’ solutions**

Unfortunately, we cannot expect the strong condition (4.8) to be sustainable for large times. This can be seen immediately with the trivial solutions (4.6) already mentioned, namely

$$v(t, x) = 0, \quad y(t, x) = \nabla p(t, x), \quad p(t, x) = p_0(x) + t g(x)$$

Indeed, it is sufficient to have a source term $G = \nabla g$, with $D^2 g(x) \leq -c \text{Id}$ (in the sense of symmetric matrices) for some positive constant $c$, to fail the strong convexity condition (4.8) in finite time. However, these trivial solutions, of both the HB and the RNSB system, can be expected to be dynamically very unstable solutions of the RNSB equations (4.2), as $\epsilon$ gets smaller and smaller. This is why it seems reasonable to look for solutions of HB system (4.3) which keep the convexity condition, at least in the large sense

$$D^2 p(t, x) \geq 0.$$  (4.14)

In the framework of Hoskins’ semigeostrophic equations [15], this condition is called the Cullen–Purser condition [6]. By analogy with the theory of hyperbolic conservation laws [1], we call (4.14) the ‘entropy condition’ (or, alternatively, the generalized Cullen–Purser condition).

Our key observation now is that any solution $y(t, x) = \nabla p(t, x)$, square integrable at each time $t$, which enjoys the ‘entropy’ condition (4.14), can be entirely recovered by the knowledge of all ‘observables’

$$f \to \int_D f(y(t, x)) \, dx,$$  (4.15)

for all continuous function $f$ with at most quadratic growth at infinity. This is a direct consequence of the rearrangement theorem 3.1. But, we have, for free, an evolution equation for all these observables, namely (4.4), which is valid for the RNSB equations (4.2), independently of both $v$ and $\epsilon$. This suggest the following concept of ‘entropy’ solution for HB system (4.3):

**Definition 4.6.** We say that $(t \to y(t, \cdot)) \in C^0(R_+, L^2(D, R^d))$ is an entropy to HB system (4.3), if, for each time $t$, $y = \nabla p$ is a map with convex potential and if

$$\frac{d}{dt} \int_D f(y(t, x)) \, dx = \int_D (\nabla f)(y(t, x)) \cdot G(t, x) \, dx,$$  (4.16)

for all $C^1$ function $f$ with $|\nabla f(y)| \leq \text{cst}|y|$.

**e) Global existence of ‘entropy’ solutions**

The global existence of entropy conditions is an easy consequence of the convergence of the multi-dimensional rearrangement scheme:

**Theorem 4.7.** As $h \to 0$, the multi-dimensional rearrangement scheme has converging subsequences. Each limit $y$ belongs to the space $C^0(R_+, L^2(D, R^d))$, has a convex potential: $y(t, \cdot) = \nabla p(t, \cdot)$ for each $t \geq 0$ and satisfies

$$\frac{d}{dt} \int_D f(y(t, x)) \, dx = \int_D (\nabla f)(y(t, x)) \cdot G(t, x) \, dx$$  (4.17)

for all smooth function $f$ such that $|\nabla f(x)| \leq (1 + |x|)\text{cst}$. This exactly means that $y$ is a global entropy solutions to HB equations (4.3).

The proof will just be sketched below (see [23] for more details). Note that the system is self-consistent, thanks to the rearrangement theorem. However, this global existence result does not imply uniqueness of solutions with respect to initial conditions.
Sketch of proof. Let us first check the consistency of the multi-dimensional rearrangement scheme (3.2) with relation (4.16). Take a smooth function \( f \). Then,

\[
\int_D f(y_{n+1}(x)) \, dx = \int_D f(\tilde{y}_{n+1}(x)) \, dx
\]

(because \( y_{n+1} \) is a rearrangement of \( \tilde{y}_{n+1} \))

\[
= \int_D f(y_n(x) + hG(nh, x)) \, dx
\]

(by definition of predictor \( \tilde{y}_{n+1} \))

\[
= \int_D f(y_n(x)) \, dx + h \int_D (\nabla f)(y_n(x)) \cdot G(nh, x) \, dx + o(h),
\]

which is an obvious time-discrete version of (4.16). Beyond consistency, because of this very relation, the approximate solutions automatically enjoy enough compactness with respect to the time variable, meanwhile their strong compactness with respect to the space variable simply follows from the convexity condition.

5. Open problems

Let us sketch two main open problems:

**Stability and singularities.** *A priori,* our global ‘entropy’ solutions of the HB model are stable with respect to initial conditions only in some special cases, such as \( d = 1 \) or \( G(x) = -x \) (where we can rely on the theory of maximal monotone operators [24]). Clearly, this needs to be extended to all cases. Moreover, strict convexity clearly breaks down in finite time for some data, but is it generically true? This is clear only for \( d = 1 \) thanks to the theory of scalar conservation laws.

**Convergence toward entropy solutions beyond singularities.** A much more challenging problem is to prove, after strict convexity breaks down, that the ‘entropy’ solutions which obey the convexity principle, correctly describe the limit of the rescaled NSB solutions in the HB regime, at least for ‘generic’ initial conditions. They may be just crude (but relevant) approximations, in some suitable sense for which a right mathematical framework has to be found. A similar, but probably even more challenging, situation occurs in shallow water theory when shock waves (‘hydraulic jumps’) appear.

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Appendix A

Let us give a comprehensive proof of theorem 4.5. On the periodic box \( \mathbb{R}^d/\mathbb{Z}^d \), it is convenient to introduce the periodic unknowns

\[
\psi(t, y) = p^s(t, y) - \frac{|y|}{2}, \quad z(t, x) = y(t, x) - x, \quad z^s(t, x) = y^s(t, x) - x, \quad (A1)
\]

We introduce the ‘relative entropy’ (or Bregman) function

\[
\eta_p(t, y, y') = p^s(t, y') - p^s(t, y) - (\nabla p^s)(t, y) \cdot (y' - y) \sim |y' - y|^2, \quad (A2)
\]

which control \( |y' - y|^2 \) because of the strong convexity assumption (4.9). Accordingly, given a weak solution \((z^s, v^s)\) to the NSB equations (4.2) and a solution \((z, v)\) of (4.3), we introduce the relative-entropy functional

\[
e(t) = \int \eta_p(t, x + z(t, x), x + z^s(t, x)) \, dx, \quad (A3)
\]
where the integral in $x$ is implicitly performed over $R^d/Z^d$ (as will be done subsequently). This functional controls the squared $L^2$ distance between $z$ and $z^\varepsilon$. We also introduce the augmented functional

$$\tilde{e}(t) = e(t) + \frac{\varepsilon}{2} \int |v^\varepsilon - v|^2 \, dx$$

(A 4)

and want to prove

$$\frac{d}{dt} \tilde{e}(t) \leq (\tilde{e}(t) + \varepsilon)c,$$

(A 5)

where $c$ depends only on the limit solution $(z, v)$ on a fixed finite time interval $[0, T]$ on which $(z, v)$ is smooth. From this estimate (A 5), we immediately deduce that $z - z^\varepsilon$ is of order $O(\sqrt{\varepsilon})$ in $L^\infty([0, T], L^2(R^d/Z^d))$. So, we are left with proving (A 5).

First, according to (A 1), we write

$$p^\varepsilon(t, y) = \frac{|y|^2}{2} + \psi(t, y),$$

where $\psi(t, y)$ is $Z^d$ periodic in $y$ and note that

$$\nabla p^\varepsilon(x + z(t, x)) = x,$$

(A 6)

by definition of the Legendre–Fenchel transform (4.10). Thus,

$$p^\varepsilon(t, x + z^\varepsilon(t, x)) - p^\varepsilon(t, x + z(t, x)) - (\nabla p^\varepsilon)(t, x + z(t, x)) \cdot (z^\varepsilon(t, x) - z(t, x))$$

$$= \psi(t, x + z^\varepsilon) - \psi(t, x + z) + \frac{|x + z^\varepsilon|^2}{2} - \frac{|x + z|^2}{2} - x \cdot (z^\varepsilon - z)$$

$$= \psi(t, x + z^\varepsilon) - \psi(t, x + z) + \frac{|z^\varepsilon|^2}{2} - \frac{|z|^2}{2}$$

and we can rewrite $e(t)$ as

$$e(t) = \int \left( \psi(t, x + z^\varepsilon) - \psi(t, x + z) + \frac{|z^\varepsilon|^2}{2} - \frac{|z|^2}{2} \right) \, dx.$$

From the NSB equations, we will use the three essential following properties:

— the energy inequality

$$\frac{\varepsilon}{2} \frac{d}{dt} \int |v^\varepsilon|^2 \, dx + \alpha \int |Dv^\varepsilon|^2 \, dx \leq \int v^\varepsilon \cdot z^\varepsilon \, dx,$$

(A 7)

— the weak formulation (obtained by formal integration by part in $x$) applied to any smooth divergence-free vector field $w(t, x)$, $Z^d$ periodic in $x$

$$\frac{\varepsilon}{2} \frac{d}{dt} \int v^\varepsilon \cdot w \, dx = \varepsilon \int (v^\varepsilon \cdot \partial_t w + (v^\varepsilon \otimes v^\varepsilon) : D w) \, dx + \alpha \int v^\varepsilon \cdot \Delta w \, dx + \int w \cdot z^\varepsilon \, dx,$$

(A 8)

— the ‘renormalized version’ (in the sense of DiPerna & Lions [20]) of the transport equation applied to any test function $r(t, x, z)$, $Z^d$ periodic in $x$ (obtained by a formal combination of integration by part in $x$ and chain rule):

$$\frac{d}{dt} \int r(t, x, z^\varepsilon(t, x)) \, dx = \int \left( (\partial_t r)(t, x, z^\varepsilon(t, x)) + (\nabla_z r)(t, x, z^\varepsilon(t, x)) \cdot (G(t, x) - v^\varepsilon(t, x)) + (\nabla_x r)(t, x, z^\varepsilon(t, x)) \cdot v^\varepsilon(t, x) \right) \, dx.$$

(A 9)

These properties follow from the combination of Leray theory of Navier–Stokes equations and the DiPerna–Lions theory of ODEs with Sobolev coefficients. For these topics, we refer to Lions’s book [16].
Let us write
\[
\frac{d}{dt} e(t) = I_1 + I_2 + I_3 + I_4,
\]
\[
I_1 = \frac{d}{dt} \int [\psi(t, x + z^e(t, x))] \, dx,
\]
\[
I_2 = -\frac{d}{dt} \int [\psi(t, x + z(t, x))] \, dx,
\]
\[
I_3 = \frac{d}{dt} \int \left[ \frac{|z^e(t, x)|^2}{2} \right] \, dx
\]
and
\[
I_4 = -\frac{d}{dt} \int \left[ \frac{|z(t, x)|^2}{2} \right] \, dx.
\]
We first get, thanks to (A 9), with \( r(t, x, z) = \psi(t, x + z) \),
\[
I_1 = \int [(\partial_t \psi)(t, x + z^e) + (\nabla \psi)(t, x + z^e) \cdot G] \, dx
\]
and, similarly,
\[
I_2 = -\int [(\partial_t \psi)(t, x + z) + (\nabla \psi)(t, x + z) \cdot G] \, dx.
\]
Thus, using the Bregman functions, respectively, corresponding to \( \partial_t \psi(t, y) \) and \( \nabla \psi(t, y) \), we can write
\[
\int [\eta_{\partial_t \psi}(t, x, x + z^e) + \eta_{\nabla \psi}(t, x + z, x + z^e) \cdot G] \, dx = I_1 + I_2 + I_5,
\]
where
\[
I_5 = -\int [(\partial_t \nabla \psi)(t, x + z(t, x)) + (D^2 \psi)(t, x + z(t, x)) \cdot G(t, x)) \cdot (z^e - z)(t, x) \, dx.
\]
Because of the structure of Bregman functions, we have
\[
I_1 + I_2 + I_5 \leq c(t)e(t),
\]
where \( e \) depends only on \( z \) (through \( \psi \) and \( G \). Let us now treat \( I_5 \). Using the duality relation (A 6),
we get
\[
x + z(t, x) + \nabla \psi(t, x + z(t, x)) = x.
\]
Differentiating in \( x \) and \( t \), we, respectively, deduce
\[
(I + (D^2 \psi)(t, x + z))(I + Dz) = I
\]
and
\[
(\partial_t \nabla \psi)(t, x + z) + (I + (D^2 \psi)(t, x + z)) \cdot \partial_t z = 0
\]
Let us rewrite HB equations (4.3) in terms of \( z(t, x) = y(t, x) - x \):
\[
\partial_t z = G - (I + Dz) \cdot v.
\]
Thus, we find
\[
(\partial_t \nabla \psi)(t, x + z) = (I + (D^2 \psi)(t, x + z)) \cdot (-G + (I + Dz) \cdot v) = -G + (D^2 \psi)(t, x + z) \cdot G + v,
\]
i.e.
\[
(\partial_t \nabla \psi)(t, x + z) + (D^2 \psi)(t, x + z) \cdot G = v - G
\]
and, accordingly, we may write
\[
I_5 = -\int (v - G) \cdot (z^e - z) \, dx.
\]
So, we have obtained so far
\[
I_1 + I_2 \leq \int (v - G) \cdot (z^e - z) \, dx + e(t)e(t).
\]
Let us now move to $I_3$ and $I_4$. Using the NSB equations, in particular (A 9) with $r(t, x, z) = |z|^2$, we find

$$I_3 = \frac{d}{dt} \int \frac{|\varepsilon v^\varepsilon(t, x)|^2}{2} \, dx = \int (G - v^\varepsilon) \cdot z^\varepsilon \, dx,$$

and similarly

$$I_4 = -\frac{d}{dt} \int \frac{|z(t, x)|^2}{2} \, dx = -\int (G - v) \cdot z \, dx.$$

So, we have obtained

$$\frac{d}{dt} \varepsilon(t) = I_1 + I_2 + I_3 + I_4 \leq \int (v - v^\varepsilon) \cdot z^\varepsilon \, dx + \varepsilon(t)c. \tag{A 10}$$

The right-hand side seems out of control because, a priori, $v^\varepsilon$ can be very large as $\varepsilon$ tends to zero. Fortunately, there is an exact compensation with the kinetic energy balance. Indeed, from the NSB equations, we deduce, combining ((A 8) and (A 7))

$$\frac{\varepsilon}{2} \frac{d}{dt} \int |v^\varepsilon - v|^2 \, dx + \alpha \int |Dv^\varepsilon|^2 \, dx + \int (v - v^\varepsilon) \cdot z^\varepsilon \, dx$$

$$\leq \varepsilon \int v \cdot \partial_t v \, dx - \varepsilon \int v^\varepsilon \cdot \partial_t v \, dx - \varepsilon \int (v^\varepsilon \otimes v^\varepsilon) \cdot Dv \, dx - \alpha \int v^\varepsilon \cdot \Delta v \, dx$$

$$\leq \varepsilon \left( \int |v^\varepsilon - v|^2 \, dx + 1 \right) c,$$

where $c$ depends only on the limit $v$ and $\alpha = O(\varepsilon)$ is assumed. Thus, combining this inequality and (A 10), we get for $\varepsilon(t)$, defined by (A 4),

$$\frac{d}{dt} \varepsilon(t) + \alpha \int |Dv^\varepsilon|^2 \, dx \leq (\varepsilon(t) + \varepsilon)c.$$

This is stronger than the desired inequality (A 5) and completes the proof.

References


