

Research



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Geometry of mass

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We study the effect of mass on geometric descriptions of gauge field theories. In an approach in which the massless theory resembles general relativity, the introduction of the mass entails non-zero torsion and the generalization to Einstein–Cartan–Sciama–Kibble theories. The relationships to pure torsion formulations (teleparallel gravity) and to higher gauge theories are also discussed.

1. Introduction

The origin and significance of mass is one of the most important topics of modern physics. Against this backdrop, the title 'geometry of mass' possesses a variety of meanings. The present contribution could have been about anti-de Sitter geometries emerging in low-energy descriptions of strongly interacting gauge field theories and the warping of these geometries due to conformal symmetry-breaking effects [1–4]. Rather differently, this contribution could have concerned the different reaction of geometry to energy–momentum sources in massive gravities [5–8]. Ultimately, as was already announced in the abstract, we will leave these two aspects aside, but elaborate on the meaning of mass in geometric duals of gauge field theories [9,10].

In this gauge-theory context, arguably the most important recent discovery under the heading 'mass' was that of a Higgs boson [11]. It is needed to make the standard model theoretically consistent—in particular, perturbatively renormalizable and weakly interacting all the way up to high energy scales—while allowing for massive weak gauge bosons and fermions. Thus, the standard model seems to capture the essence of how electroweak symmetry breaking looks around the electroweak scale and below. The standard model, however, only parametrizes the origin of mass, but does not explain it. Unfortunately, no smoking guns have been found that would point towards

the fundamental mechanism. Moreover, by far the major part of the mass we observe around us every day is not created at the electroweak scale anyhow, but is generated dynamically by the strong nuclear force, as described by the gauge theory of quantum chromodynamics. Furthermore, none of the other scalar excitations found in nature to date were elementary, which, to the contrary, is assumed for the standard-model Higgs. In any case, in particle physics, the phenomenon of mass occurs in connection to non-Abelian gauge field theories, and here we are interested in the meaning of these masses. We would like to see what their effect is in a geometric setting. Furthermore, the geometric formulation is based on gauge-invariant variables, which makes the connection to gauge-invariant quantities, i.e. observables, more direct.

The rest of the paper is organized as follows. Section 2 is concerned with the instructive three-dimensional case, in particular, elaborating on the difference between the presence and absence of a mass. Section 3 treats the four-dimensional case, most relevant to our home space–time. Section 4 puts the results in a larger context and concludes this paper.

2. Three dimensions

(a) Massless

Let us begin with three-dimensional massless Yang–Mills theory. In the first-order formalism [12], the Lagrangian density reads

$$\mathcal{L}_0 = -\frac{e^2}{2} E_\mu^a E_\mu^a + \frac{i}{2} \epsilon^{\mu\nu\lambda} E_\mu^a F_{\nu\lambda}^a, \quad (2.1)$$

where e stands for the gauge coupling, E_μ^a for the dual field strength, $\epsilon^{\mu\nu\lambda}$ for the completely antisymmetric tensor, $F_{\mu\nu} = i[D_\mu, D_\nu]$ for the field-strength tensor and D_μ for the covariant derivative belonging to the gauge field A_μ . For the sake of concreteness we will work in Euclidean space, but adopting a different metric signature is straightforward. Integrating out the dual field strength E_μ^a amounts to replacing it by its saddle-point value,

$$e^2 \check{E}_\mu^a = \frac{i}{2} \epsilon^{\mu\nu\lambda} F_{\nu\lambda}^a, \quad (2.2)$$

and yields the Lagrangian density in its better-known second-order form,

$$\mathcal{L}_0 \mapsto -\frac{1}{4e^2} F_{\mu\nu}^a F_{\mu\nu}^a. \quad (2.3)$$

Here, there are no additional physically relevant corrections due to fluctuations, because the Lagrangian is at most quadratic in the field E_μ^a , and the quadratic term is independent of any other dynamic field.

This is different when eliminating the gauge connection A_λ^a from (2.1). While the Lagrangian density is still only quadratic in this field, the quadratic term depends on the other dynamical field. Therefore, there will be fluctuation corrections, which we will discuss below.

Up to fluctuations (see below), integrating out the gauge connection amounts to evaluating the action $S_0 = \int d^3x \mathcal{L}_0$ at the corresponding saddle point. The saddle-point condition is obtained by equating the variation of the action S_0 with respect to the gauge field A_λ^a , $\delta_{A_\lambda^a} S_0$, to zero and is given by

$$\epsilon^{\mu\nu\lambda} D_\mu^{ab}(\check{A}) E_\nu^b = 0. \quad (2.4)$$

In order to exploit this condition further, we expand the covariant derivative of the dual field strength in terms of the dual field strength itself,

$$D_\mu^{ab}(\check{A}) E_\nu^b = \Gamma_{\mu}{}^{\kappa}{}_{\nu} E_\kappa^a, \quad (2.5)$$

where we have chosen an $SU(2)$ gauge group, where the different components of the dual field strength span the entire colour space. (It represents a dreibein.) Thus, the expansion coefficients

are unique. (For larger gauge groups see §2a(i).) Then the saddle-point condition implies that the expansion coefficients are symmetric in their two lower indices,

$$\epsilon^{\mu\nu\lambda}\Gamma_{\mu}{}^{\kappa}{}_{\nu}=0, \quad (2.6)$$

i.e. they are free of torsion. Indeed, these connection coefficients coincide with the Christoffel symbols

$$2\Gamma_{\mu}{}^{\kappa}{}_{\nu}=g^{\kappa\lambda}(\partial_{\mu}g_{\nu\lambda}+\partial_{\nu}g_{\mu\lambda}-\partial_{\lambda}g_{\mu\nu}) \quad (2.7)$$

for the metric

$$g_{\mu\nu}:=E_{\mu}^a E_{\nu}^a. \quad (2.8)$$

With the help of the connection coefficients, we can define the covariant derivative

$$\nabla_{\mu}{}^{\kappa}{}_{\nu}:=\partial_{\mu}\delta_{\nu}^{\kappa}+\Gamma_{\mu}{}^{\kappa}{}_{\nu} \quad (2.9)$$

and subsequently the Riemann tensor

$$R^{\kappa}{}_{\lambda\mu\nu}:=[\nabla_{\mu},\nabla_{\nu}]_{\lambda}^{\kappa}. \quad (2.10)$$

It is linked to the field strength tensor at the saddle point by

$$F_{\mu\nu}^a(\check{A})=\frac{1}{2}\epsilon^{abc}E^{b\lambda}E_{\kappa}^cR^{\kappa}{}_{\lambda\mu\nu}. \quad (2.11)$$

Putting this last result and the definition (2.8) of the metric $g_{\mu\nu}$ into the Lagrangian \mathcal{L}_0 , we obtain the classically equivalent Lagrangian [13]

$$\mathcal{L}_0\mapsto-\frac{e^2}{2}g_{\mu\mu}+\frac{i}{2}\sqrt{g}R, \quad (2.12)$$

where g stands for the determinant of $g_{\mu\nu}$ and R for the Ricci scalar. The second addend is the Einstein–Hilbert term for three-dimensional gravity, a topological field theory without any propagating degrees of freedom. The first addend, however, is background-dependent. It consists of the ‘trace’ of the dynamical metric $g_{\mu\nu}$ constructed with the (here, by our choice) fixed flat Euclidean inverse metric $\delta^{\mu\nu}$. Thus, this classically equivalent theory propagates three transverse degrees of freedom like the original Yang–Mills formulation. The high degree of symmetry (including diffeomorphism invariance) of the second addend, which there bars any propagating degrees of freedom, is inherited from the symmetries of the EF term in the original first-order Lagrangian (2.1): the entire Lagrangian is invariant under the simultaneous gauge transformations

$$\delta A_{\mu}^a=D_{\mu}^{ab}\phi^b \quad \text{and} \quad \delta E_{\mu}^a=f^{abc}E_{\mu}^b\phi^c. \quad (2.13)$$

Additionally, due to the Bianchi identities, the second term separately is invariant under the shift

$$\delta A_{\mu}^a=0 \quad \text{and} \quad \delta E_{\mu}^a=D_{\mu}^{ab}(A)\theta^b. \quad (2.14)$$

Certain linear combinations of these two transformations amount to general diffeomorphism,

$$\phi^a=n^{\lambda}A_{\lambda}^a \quad \text{and} \quad \theta^a=n^{\lambda}E_{\lambda}^a \Rightarrow \delta E_{\mu}^a=n^{\lambda}\partial_{\lambda}E_{\mu}^a+E_{\lambda}^a\partial_{\mu}n^{\lambda}. \quad (2.15)$$

Integrating out the gauge connection also gives rise to a fluctuation determinant

$$E^{-3/2} \propto g^{-3/4} \quad (2.16)$$

on the ground floor of the path integral arising from the term $\epsilon^{\mu\nu\lambda}\epsilon^{abc}E_\mu^a A_\nu^b A_\lambda^c$ in the action. Here E stands for the determinant of the dreibein. An additional Jacobian arises when we change variables from the gauge-dependent vielbein E_μ^a to the gauge-invariant metric $g_{\mu\nu}$,

$$d^9 E_\mu^a = d^3 \Omega d^6 g_{\mu\nu} g^{-1/2}. \quad (2.17)$$

The metric carries the information about the orientation of the dreibein components relative to one another and about their length, but not about their absolute orientation in space, which is encoded in the angular part $d^3 \Omega$, for example, the Euler angles.

(i) Different gauge groups

For a larger gauge group than $SU(2)$, say $SU(N)$, for the sake of concreteness, the definition (2.5) of the connection coefficients is not unique, as the dreibein E_μ^a does not span the entire colour space. Therefore, we must extend the basis correspondingly,

$$D_\mu^{ab}(\check{A})E_N^b = \Gamma_\mu|_N^K E_K^a, \quad (2.18)$$

where the space–time indices (capital letters) count from 1 to $N^2 - 1$ as do the group indices (small letters). In fact, this completed basis decomposes into subgroups of $SU(N)$,

$$E_{\mu_1}^a, E_{\{\mu_1\mu_2\}}^a, \dots, E_{\{\mu_1 \dots \mu_{N-1}\}}^a, \quad (2.19)$$

where the accolades denote not only complete symmetrization but also complete tracelessness of any two indices [14]. These are exactly the $N^2 - 1$ needed basis vectors. The basis vectors belonging to different subgroups correspond to different spins.

(ii) Limit of vanishing coupling

The perturbative expansion for small coupling e of the theory described by the classically equivalent Lagrangian (2.12) is around a topological field theory, i.e. three-dimensional Einstein–Hilbert gravity. In particular, topological or not, the theory is generally covariant insofar as it is not formulated over a fixed background. This feature is already discernible in the first-order Lagrangian (2.1), where the remaining second addend makes no reference to any background metric. In the second-order Yang–Mills Lagrangian (2.3), the zero-coupling limit enforces a vanishing field strength $F_{\mu\nu}^a$. This implies that only pure-gauge configurations are allowed for the gauge field A_μ^a , which thus does not carry any propagating degrees of freedom.

This is consistent with the observation that three-dimensional gravity is classically equivalent to Chern–Simons theory [15], which also does not need any background metric for its formulation. The corresponding classical equations of motion for the Chern–Simons theory impose that the field tensor for the Chern–Simons connection vanish. Consequently, there are no propagating degrees of freedom. (This also holds in the presence of a cosmological constant; see [15].) In this context, the extension to larger gauge groups discussed in §2a(i) leads to the concept of higher-spin gravity including generalized three-dimensional black holes [16] and, more generally, to higher-spin field theories. (For a review see [17].)

(b) Massive

Now let us turn to the massive case. (Here, we will only treat a bare Proca mass term, which can also be seen as a nonlinear sigma model in unitary gauge. Accordingly, we also do not treat what is usually referred to as spontaneous symmetry breaking and the Higgs mechanism. For these extensions and also for the extension to non-simple gauge groups, see [9,10].) The Lagrangian is

obtained by adding the mass term for the Yang–Mills field, which, following [9], we express with the help of a Lagrange multiplier field (the dual of a two-form) C_μ^a ,

$$\mathcal{L} := \mathcal{L}_0 - \frac{m^2}{2} A_\mu^a A_\mu^a \rightarrow \quad (2.20)$$

$$\rightarrow \mathcal{L}_0 - \frac{1}{2} C_\mu^a C_\mu^a + im C_\mu^a A_\mu^a. \quad (2.21)$$

We proceed as before and eliminate the gauge field A_λ^a by means of its saddle-point condition

$$\epsilon^{\mu\nu\lambda} D_\mu^{ab}(\check{A}) E_\nu^b = m C_\lambda^a. \quad (2.22)$$

With the same definition (2.5) for the connection coefficients as before, the saddle-point condition becomes

$$\epsilon^{\mu\nu\lambda} \Gamma_\mu|_\nu E_\kappa^a = m C_\lambda^a. \quad (2.23)$$

The antisymmetric part of the connection coefficient does not vanish any more (unless $C_\lambda^a = 0$). This is due to the three additional degrees of freedom propagated by the massive theory. As a consequence, the connection carries torsion. Introducing the inverse of the dreibein,

$$E_\kappa^a E^{a\lambda} = \delta_\kappa^\lambda, \quad (2.24)$$

the torsion tensor is given by

$$S_\mu|_\nu^\kappa := \Gamma_{[\mu}^\kappa|_{\nu]} = \frac{1}{2} m C_\lambda^a E^{a\kappa} \epsilon^\lambda_{\mu\nu}. \quad (2.25)$$

After defining the contorsion tensor

$$K_{\mu\nu|\kappa} = -S_{\mu\nu|\kappa} - S_{\kappa\mu|\nu} + S_{\nu\kappa|\mu} = -K_{\mu\kappa|\nu}, \quad (2.26)$$

we can split the connection coefficients into the latter and their torsion-free part $\mathring{\Gamma}_\mu|_\nu^\kappa$,

$$\Gamma_\mu|_\nu^\kappa = \mathring{\Gamma}_\mu|_\nu^\kappa - K_{\mu\nu}^\kappa. \quad (2.27)$$

If we define the covariant derivative as in (2.9), the Riemann tensor (2.10) decomposes as follows:

$$R^\mu{}_{\nu\kappa\lambda} = \mathring{R}^\mu{}_{\nu\kappa\lambda} + (-\mathring{\nabla}_\lambda K_\kappa|_\nu^\mu + K_\lambda|_\rho^\mu K_\kappa|_\nu^\rho) - (\kappa \leftrightarrow \lambda), \quad (2.28)$$

where $\mathring{\nabla}_\lambda$ is the covariant derivative constructed with the torsion-free connection coefficients and $\mathring{R}^\mu{}_{\nu\kappa\lambda}$ is the Riemann tensor computed from it. For the Ricci scalar this means

$$R = \mathring{R} - 4\mathring{\nabla}_\mu S^\mu|_\nu^\nu + K_{\nu\kappa|\mu} K^{\mu\nu|\kappa} - 4S_\mu|_\nu^\nu S^\mu|_\kappa^\kappa, \quad (2.29)$$

where \mathring{R} is the Ricci scalar for the torsion-free covariant derivative. In terms of the Ricci scalar R from (2.29) and $g_{\mu\nu}$ from (2.8), the classically equivalent Lagrangian density looks just like (2.12) plus the mass term from (2.21). In terms of C_μ^a , the three additional degrees of freedom can be expressed as

$$\hat{\vec{C}}^a \cdot \vec{E}^a, \quad \hat{\vec{C}}^a \times \vec{E}^a, \quad \vec{C}^a \cdot \vec{C}^a, \quad \text{where } \hat{\vec{C}}^a = \frac{\vec{C}^a}{\sqrt{\vec{C}^a \cdot \vec{C}^a}}. \quad (2.30)$$

The appearance of the new degrees of freedom can also be traced conveniently in the expression for the spin connection ω_μ^{bc} , which, up to a duality transformation in colour space, coincides with the gauge field on the saddle point,

$$\check{A}_\mu^a \epsilon^{abc} = \omega_\mu^{bc} = E_\nu^b (\partial_\mu E^{c\nu} + E^{c\kappa} \Gamma_\kappa|_\mu^\nu) \quad (2.31)$$

$$= E_\nu^b [\partial_\mu E^{c\nu} + E^{c\kappa} (\mathring{\Gamma}_\kappa|_\mu^\nu - K_\kappa|_\mu^\nu)]. \quad (2.32)$$

The connection part picks up a new piece in the form of the contorsion tensor.

3. Four dimensions

Our physical world appears to be four-dimensional. Hence, after the instructive study of the three-dimensional case, let us proceed to four space–time dimensions, where we keep the mass term right away. Here, the first-order form of the Lagrangian density involves a two-form instead of the one-form field $B_{\mu\nu}^a$ and is given by

$$\mathcal{L}_0 := -\frac{e^2}{4} B_{\mu\nu}^a B_{\mu\nu}^a + \frac{i}{4} \epsilon^{\mu\nu\kappa\lambda} B_{\mu\nu}^a F_{\kappa\lambda}^a - \frac{1}{2} C_\mu^a C_\mu^a + i m C_\mu^a A_\mu^a. \quad (3.1)$$

The BF term again exhibits an enhanced symmetry including invariance under diffeomorphisms: it and the BB term are invariant under the gauge transformations

$$\delta A_\mu^a = D_\mu^{ab} \phi^b \quad \text{and} \quad \delta B_{\mu\nu}^a = f^{abc} B_{\mu\nu}^b \phi^c, \quad (3.2)$$

while the BF term is also invariant under

$$\delta A_\mu^a = 0 \quad \text{and} \quad \delta B_{\mu\nu}^a = D_{[\mu}^{ab} (A) \theta_{\nu]}^b, \quad (3.3)$$

owing to the Bianchi identities. Special linear combinations amount to a general diffeomorphism,

$$\phi^a = n^\lambda A_\lambda^a \quad \text{and} \quad \theta_v^a = n^\lambda B_{\lambda\nu}^a \Rightarrow \delta B_{\mu\nu}^a = n^\lambda \partial_\lambda B_{\mu\nu}^a + B_{\lambda\nu}^a \partial_\mu n^\lambda + B_{\mu\lambda}^a \partial_\nu n^\lambda. \quad (3.4)$$

Indeed, the BF term alone is, for example, the basis for topological BF gravity and related approaches to four-dimensional gravity (e.g. [18,19]).

The saddle-point condition for integrating out the gauge field A_ν^a reads

$$\frac{1}{2} \epsilon^{\kappa\lambda\mu\nu} D_\mu^{ab} (\check{A}) B_{\kappa\lambda}^b = m C_\nu^a. \quad (3.5)$$

There are different ways for making use of this relation (e.g. [20]), but we would like to stay as close as possible to the procedure in the three-dimensional case, although some modifications remain necessary. We choose to interpret the two-form as a generalized vielbein (see [21] for another such example) and assume that the gauge group is six-dimensional (e.g. $SO(4)$) such that the vielbein forms a complete basis, without having to enlarge it as in §2a(i). Accordingly, as above, we expand the covariant derivative of this vielbein in the basis spanned by the vielbein,

$$D_\mu^{ab} (\check{A}) B_{\kappa\lambda}^b = \gamma_\mu |_{\kappa\lambda}^{\rho\sigma} B_{\rho\sigma}^a. \quad (3.6)$$

The expansion coefficients are antisymmetric in their last two upper, respectively, lower, indices,

$$\gamma_\mu |_{\kappa\lambda}^{\rho\sigma} = \gamma_\mu |_{\kappa\lambda}^{[\rho\sigma]} = \gamma_\mu |_{[\kappa\lambda]}^{\rho\sigma}. \quad (3.7)$$

By the choice of a six-dimensional gauge group, we have a complete basis right away. Still, the situation is akin to the one in equation (2.18); there is a disparity between the first index and the (pairs of) remaining indices, again related to the use of non-standard vielbeinen. With the above expansion, the saddle-point condition becomes

$$\epsilon^{\kappa\lambda\mu\nu} \gamma_\mu |_{\kappa\lambda}^{\rho\sigma} B_{\rho\sigma}^a = m C_\nu^a. \quad (3.8)$$

As it forms a complete basis, we can define the inverse $B^{a\kappa\lambda}$ of the vielbein through

$$B_{\rho\sigma}^a B^{a\alpha\beta} = \frac{1}{2} \delta_{\rho\sigma}^{\alpha\beta}, \quad (3.9)$$

where $\delta_{\rho\sigma}^{\alpha\beta}$ stands for a generalized Kronecker delta symbol, which equals +1 if the values of the top indices are all distinct and an even permutation of the bottom ones, −1 if they are distinct and an odd permutation, and 0 in all other cases. Then the contraction of the saddle-point condition with said inverse leads to

$$\epsilon^{\kappa\lambda\mu\nu} \gamma_\mu |_{\kappa\lambda}^{\alpha\beta} = m C_\nu^a B^{a\alpha\beta}. \quad (3.10)$$

Consequently, the fully antisymmetric part $\gamma_{[\mu} l_{\kappa\lambda]}^{\alpha\beta}$ vanishes for $m=0$, and generally does not for $m \neq 0$. (In the language chosen in [20] there was already an antisymmetric part for $m=0$.)

We can again define a covariant derivative

$$\nabla_\mu l_J^K := \partial_\mu \delta_J^K + \gamma_\mu l_J^K, \quad (3.11)$$

where we have adopted meta-indices (capital letters) in the spirit of equation (2.18); and from there a Riemann tensor,

$$R^K_{J\mu\nu} := [\nabla_\mu, \nabla_\nu] l_J^K. \quad (3.12)$$

The Riemann tensor and the field-strength tensor are related by

$$cF_{\mu\nu}^c = f^{abc} B_K^a B^b l_J^K R^K_{J\mu\nu}, \quad (3.13)$$

where $f^{abc} f^{dbc} = c\delta_d^a$. Therefore, the BF term becomes

$$\frac{i}{4} \epsilon^{\kappa\lambda\mu\nu} B_{\kappa\lambda}^a \tilde{F}_{\mu\nu}^a = \frac{i}{4} c f^{abc} B_K^a B^b l_J^K R^K_{J\mu\nu} \epsilon^{\mu\nu\kappa\lambda} B_{\kappa\lambda}^c = \frac{1}{2} c f^{abc} B_K^a B^b l_J^K R^K_{J\mu\nu} \tilde{B}^{c\mu\nu}, \quad (3.14)$$

where we have introduced the dual

$$\tilde{B}^{a\mu\nu} = \frac{i}{2} \epsilon^{\mu\nu\kappa\lambda} B_{\kappa\lambda}^a \quad (3.15)$$

in the last step. The similarity to the EF term in the three-dimensional case becomes more apparent by rewriting the latter as

$$\frac{i}{2} \epsilon^{\mu\nu\lambda} E_\mu^a F_{\nu\lambda}^a = \frac{i}{4} \epsilon^{abc} E^{a\lambda} E_\kappa^b R^\kappa_{\lambda\mu\nu} \epsilon^{\mu\nu\rho} E_\rho^c = \frac{1}{2} \epsilon^{abc} E^{a\lambda} E_\kappa^b R^\kappa_{\lambda\mu\nu} \tilde{E}^{c\mu\nu}, \quad (3.16)$$

where

$$\tilde{E}^{a\mu\nu} = \frac{i}{2} \epsilon^{\mu\nu\kappa} E_\kappa^a. \quad (3.17)$$

In fact, this result can be expressed in the form of (3.14) for any dimension. In three dimensions it turned out to have the most straightforward interpretation.

With the definition

$$B_{\mu\nu}^a B_{\kappa\lambda}^a \equiv B_J^a B_K^a =: g_{JK} \quad (3.18)$$

for the metric, the BB term becomes

$$-\frac{e^2}{4} B_{\mu\nu}^a B_{\mu\nu}^a \equiv -\frac{e^2}{4} B_J^a B_J^a =: -\frac{e^2}{4} g_{JJ} \quad (3.19)$$

and is background-dependent as before.

Again integrating out the gauge field also entails a fluctuation determinant,

$$B^{-3} = g^{-3/2}, \quad (3.20)$$

where B stands for the determinant of B_J^a and g for that of g_{JK} .

4. Discussion

At non-zero mass, the above geometric descriptions develop an antisymmetric part of the connection. In three dimensions, this antisymmetric part is standard torsion, whereas, in higher dimensions, the interpretation becomes more involved. The fact that the longitudinal degrees of freedom of the gauge bosons are encoded in non-symmetric components of the connection remains.

Standard Einstein–Hilbert gravity requires a symmetric connection and the metric is the fundamental variable. These requirements are softened in Einstein–Cartan–Sciama–Kibble theory [22–27], where the metric and the torsion are independent degrees of freedom. Hence, the above investigation is naturally embedded in its framework. Usually torsion couples to the intrinsic angular momentum of matter. In this study, it appears due to the presence of mass.

In Einstein–Cartan–Sciama–Kibble theory, curvature and torsion are regarded as different gravitational degrees of freedom. General relativity works alone with curvature but without torsion, but there exists an equivalent formulation of gravity that works alone with torsion but without curvature.¹ This is the teleparallel [31] equivalent of general relativity. (For a review see e.g. [32].) The choice between the two formulations can be made at the level of the connection. As we have seen, for example, in our expression for the spin connection (2.32), the general connection coefficients decompose into a symmetric and an antisymmetric part. The symmetric coefficients are the Christoffel symbols and generally encode curvature. The antisymmetric part is the contorsion tensor. By imposing the vanishing of the latter, we recover general relativity. If we limit the symmetric part of the connection to be free of curvature—i.e. it only encodes inertial effects like in special relativity—but allow for non-zero contorsion, we are adopting the so-called Weitzenböck connection. Then the field strength for teleparallel gravity is exactly the torsion tensor. The Lagrangian density is quadratic in this field strength,

$$\mathcal{L}_{\parallel} := K_{\nu}{}^{\mu} K_{\nu}{}^{\rho}{}_{\mu} - K_{\rho}{}^{\mu} K_{\rho}{}^{\nu}{}_{\mu}, \quad (4.1)$$

here expressed with the contorsion tensor for the sake of brevity. The corresponding gauge potential is the translational gauge field, hence the name teleparallel. The previous Lagrangian density coincides with the Møller Lagrangian

$$\mathcal{L}_{\text{M}} := \dot{\nabla}_{\mu} e^{j\nu} \dot{\nabla}_{\nu} e^{\mu}_j - \dot{\nabla}_{\mu} e^{i\mu} \dot{\nabla}_{\nu} e^{\nu}_i, \quad (4.2)$$

here in the first-order form and where e^j_{μ} stands for the vielbein and $\dot{\nabla}_{\mu}$ for the Weitzenböck connection. The Møller Lagrangian differs from the Einstein–Hilbert term by a total derivative. Here the equivalence between the theories is most manifest. Hence, there exists a framework based entirely on torsion that one could use to study the effect of mass, which above led to the presence of torsion in an otherwise curvature-based description. In fact, every detuning of the relative factors between the differently contracted torsion tensors in (4.1) leads to the propagation of additional degrees of freedom, and, as we have seen, such detuning contributions are introduced by the mass term. Moreover, the teleparallel description is an Abelian gauge theory of the translation group (we have a Lagrangian quadratic in the field strength and not linear in the curvature like the Einstein–Hilbert action) of the translational potential and as such much closer to the theory we wanted to analyse in a geometric setting.

Another unifying viewpoint comes from a somewhat unexpected direction, higher gauge theory: above, we described standard gauge field theories in a geometric setting. In three dimensions, we found a clear-cut link of mass to torsion, and torsion is the principal ingredient of teleparallel gravity. In four (and more) dimensions, the situation is slightly less straightforward, but there gauge theory is related (at zero coupling) to *BF* theory. Exactly these two theories, teleparallel gravity [33] and *BF* theory in four dimensions, can both be described as higher gauge theories. In a nutshell, higher gauge theory is a generalization of gauge theory that describes parallel transport not only for point particles but also for extended objects. (For a review see [34].) Among other constructs, higher gauge theory involves two-groups and two-connections. In the case of the *BF* theory, the two-connection is formed by the one-form (gauge field) A (from which we calculate the field strength) and the two-form B . In the case of teleparallel gravity, the two-connection is made up of the Weitzenböck connection and a coframe field [33] linked to the aforementioned gauge field for the translation group.

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¹There also exists a discretized analogue for the correspondence between curvature and torsion, the duality between dislocations and disclinations in crystals [28–30].

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