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A model of adaptive decision-making from representation of information environment by quantum fields

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We present the mathematical model of decision-making (DM) of agents acting in a complex and uncertain environment (combining huge variety of economical, financial, behavioural and geopolitical factors). To describe interaction of agents with it, we apply the formalism of quantum field theory (QTF). Quantum fields are a purely informational nature. The QFT model can be treated as a far relative of the expected utility theory, where the role of utility is played by adaptivity to an environment (bath). However, this sort of utility–adaptivity cannot be represented simply as a numerical function. The operator representation in Hilbert space is used and adaptivity is described as in quantum dynamics. We are especially interested in stabilization of solutions for sufficiently large time. The outputs of this stabilization process, probabilities for possible choices, are treated in the framework of classical DM. To connect classical and quantum DM, we appeal to Quantum Bayesianism. We demonstrate the quantum-like interference effect in DM, which is exhibited

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as a violation of the formula of total probability, and hence the classical Bayesian inference scheme.

This article is part of the themed issue 'Second quantum revolution: foundational questions'.

1. Introduction

This paper is devoted to applications of the mathematical formalism of quantum theory to the modelling of decision-making (henceforth denoted as 'DM') in a context where there exists deep uncertainty[1] about both the possible actions of other decision-makers and the surrounding complex *information environment*. This information environment can include belief states of decision-makers, memory recollections and external economic (or also financial, social and geopolitical) contexts.[2] In economics, the information environment is typically formalized via the introduction of the notions of (i) public (non-scarce) and private (scarce) information (flows); and the notions of (ii) produced information (a dataset which needs buying for instance) and emergent information (time which reveals information for instance). The economics and finance literature has published research on how one can measure the economic value of information. The notion of 'private information price of risk' for instance measures how private information triggers changes in the price of risk within an asset pricing context [2,3]. The idea of 'degrees of information' is sometimes also considered such as 'first-order' information (which ties the information to an event) and 'higher-order' information (which refers to the information itself (for instance what is the source of the information)). The notion of 'common knowledge' which we briefly mention below is an example of such higher-order information. For a detailed discussion of many of the basic concepts (see [4, p. 171 and p. 224]).

The idea of 'information environment' is also used in physics and very often the terms 'bath' or 'reservoir' are used. In a social science adapted context (starting from physics), we can then speak of 'mental bath' or 'mental reservoir'.

In the model presented here in this paper, agents are not aiming to maximize their utility, but they instead adapt their behaviour to information gained from other agents and the environment. To describe this process of adaptivity, we adopt the mathematical apparatus of quantum field theory (QFT). In particular, the information environment is represented mathematically by quantum fields. We call this approach *QFT-inspired*. It has to be sharply distinguished from a variety of applications where genuine quantum physics is applied to cognition and DM.

The adaptive approach which we will introduce is more realistic compared to the utility-maximizing stance used in basic economics. However, the ensuing formalism is much more demanding. As is well known, from a basic economics perspective, the usual assumption made is that in the absence of any arbitrage[3] opportunities, there should exist a so-called budget feasible plan which cannot be more preferred by any other budget feasible plan [6].

From the brief discussion so far, it is intuitive that our DM model will differ crucially from the classical expected utility theory introduced by von Neumann & Morgenstern [7]. Nevertheless, the QFT-inspired adaptive model can be treated as a far relative of utility-based models of DM, including expected utility. The role of utility is played by the degree of adaptivity to the surrounding (informational) environment, i.e. the mental bath. However, it is impossible to encode 'environment-adaptivity' by a numerical utility-function. As we will show, it is encoded by the dynamics of quantum decision operators.

[1]We will discuss below what we mean with 'deep uncertainty'.

[2]The notion 'state of the world' used in economics (and DM) and financial asset pricing can be considered as the counterpart of these physical notions [1].

[3]An arbitrage opportunity can be formally defined [5]. Essentially, an arbitrage opportunity arises when no investment needs to be made to obtain a positive cash flow. This is akin to a situation where a positive return on an investment exists without the investment itself having any risk of its own. From a risk–reward perspective, there is a positive reward for no risk.

Within the mainstream literature, in cognitive psychology and behavioural economics, several models have been developed to measure certain effects such as 'order' effects and 'conjunction' and 'disjunction' effects. Those are effects which have been repeatedly observed in experimental laboratory settings and some of those effects have relevance to some of the important paradoxes which have affected the integrity of the axiomatic structure of key expected utility models (such as the von Neumann–Morgenstern expected utility model (Allais' paradox; see [8]) and the Savage expected utility model (Ellsberg paradox, see [9]). See also Machina [10]). A very rich literature has emerged and we must mention the work of Tversky & Kahneman [11] and Tversky & Shafir [12]. However, we need to note that those effects are now also modelled with a so-called 'quantum-like' approach. In essence, this approach uses formalisms from quantum mechanics to augment the modelling of non-physics problems, such as cognition. For representative work in this area, see [13–44] and references therein.

Finally, the quantum-like approach has recently been successfully applied to the theory of common knowledge and the modelling of violations of Aumann's theorem about the impossibility to agree on disagreeing [36,37].

As was already pointed out, we are interested in modelling the DM process as being adaptive to the information environment. An agent $\mathcal{G}$ is not driven to maximize his utility, but he rather tries to adapt his behaviour to information which he is gaining (during the DM process) from his environment $\mathcal{R}$.

There are two well-known main approaches to the modelling of quantum adaptive dynamics. One approach is based on the theory of *open quantum systems.* Here, the state of an agent is extracted from the general state of the compound system 'agent+environment' and the dynamics are reduced (by using the trace operation) to the state space of this agent (the environment is encoded in the operator coefficients of the quantum master equation).[4] Another approach is based on the study of the general dynamics of the state of the compound system, 'agent+environment', and then averages and probabilities corresponding to the agents' decisions are extracted from the complete dynamical state [45]. In the latter approach, an environment is represented as a quantum field. In this paper, we proceed with this QFT-inspired approach by using the mathematical methods developed in recent papers by Bagarello [19–21] and in the monograph [18].

The main advantage of the open quantum system approach is the reduction of the state dimension to the dimension of the agent's state. The main problem is that an environment appears in a very formal encoding and it is not easy to extract its features from the coefficients of the quantum master equation. Thus, *the model is phenomenological*.[5]

The QFT-inspired approach is more difficult analytically, because the scenarios of evolution are presented by taking into account all degrees of freedom of the environment.[6] It can be difficult to construct analytical solutions of the dynamical equations for the state evolution, even if in many relevant cases this can be done (see §§3 and 4). Sometimes, however, numerical methods are really what is needed. This is the case, for instance, of nonlinear models, which will not play any role in this paper. Among the advantages of the QFT-inspired approach, we can point to the possibility of describing dynamics in terms of solely pure states.

At the beginning of this paper, we mentioned the idea of 'deep uncertainty'. From the quantum information point of view [46] a pure state represents the maximally available information about the context (in our case the context of DM). However, the availability of maximal information does not mean the resolution of uncertainty. We say that the uncertainty can be very deep. The pure state representation means that maximal information (about the context) is encoded in it. The most tricky point of quantum information theory (and, in fact, quantum theory in general) is that there exist states of compound systems, e.g. 'agent+environment', which are pure. This means they encode maximal information, but the states of subsystems, e.g. the state of the agent,

[4]This approach was used for a wide class of problems in DM, psychology, politics and biology [14–17,32,38].

[5]Another complication is that it is impossible to represent the quantum adaptive dynamics as dynamics of pure states. The impact of the environment can transfer a pure initial state into a mixed quantum state (represented by a density operator).

[6]Such an environment contains both the complex internal mental representations of agents as well as, e.g. the market situation or the current state of the political arena.

are not pure (i.e. they do not encode maximally available information about the situation). Such states of compound systems are called *entangled*. Entanglement is nowadays considered as the main distinguishing feature of quantum theory.

Let us now come back to the objective of this paper. We will study a very general DM model represented in the form of a game played by two players $\mathcal{G}_1$ and $\mathcal{G}_2$ interacting with two environments, $\mathcal{R}_1$ and $\mathcal{R}_2$. Such players can be agents of a market, e.g. corporations, traders of the financial market, political parties. A decision is taken only after the interactions $\mathcal{G}_j \leftrightarrow \mathcal{R}_j$ and $\mathcal{G}_1 \leftrightarrow \mathcal{G}_2$ are considered. The reflections of the agents generated by the interactions modify the agents' mental states. Thus, as was already emphasized, the key notion of such a DM model is not *utility*, but rather the *interactions between agents and their environments and the adaptivity to their impact*.[7]

The QFT-inspired model for this sort of games involving DM under uncertainty (including the uncertainty generated by a complex surrounding environment (be it mental, economic, social, etc.)) was considered in papers by Bagarello [19,20]. In these papers, one of the main characteristics of the analysis was that, at the beginning of the process of DM $t = 0$, each agent was in a *sharp state*. By this we mean that each agent knew, at $t = 0$, exactly which was her own choice. Thus, initially, an agent was able to resolve her internal mental uncertainty. Then, in the process of DM, her certainty was destroyed as a result of reflections generated by interactions with another agent and the environment.

Of course, this initial certainty of the agent's state is a strongly simplifying assumption which does not match reality (e.g. when we consider traders in the financial market).

In this paper, we present a more realistic model by considering the possibility that, initially, an agent is also being in a state of uncertainty which is mathematically represented as a superposition of quantum(-like) pure states corresponding to the concrete choices. Such a superposition encodes a very deep uncertainty which cannot be modelled in the framework of classical probability (CP) [27,35]. The superposition of decisions induces a kind of interference between possible decisions of agents (see §4). In probabilistic terms, this interference is visible as a violation of *the classical formula of total probability:* it is perturbed by an additional term. We will see that, in comparison with [19–21], the dynamical behaviour of *the decision functions* (DFs; see below) changes drastically. In particular, a sort of overall noise appears because of the presence of some interference effects in the system.

2. Why quantum(-like)?

The original general idea of Bagarello [18] was to formally use creation and annihilation operators, the basic building blocks of QFT, in macroscopic systems so one can possibly consider dynamical systems. These operators can be used to represent very complex dynamical processes composed of elementary acts of creation and annihilation of new states of systems. These states can be of any nature: physical, mental, biological. This is the minimal interpretation of the QFT-inspired approach. From this formal viewpoint, the application of the mathematical formalism of quantum theory is just a matter of convenience, and in fact it proved to be useful in several applications in different contexts [18–21].

In our concrete model, the states under consideration are possible strategies of players $\mathcal{G}_1$ and $\mathcal{G}_2$. *The creation and annihilation operators are explored to model the process of how players reflect upon their strategies*. These reflections are composed of elementary acts of transition of the player's mental state from one strategy to another ('annihilation' of the previously chosen strategy and 'creation' of the opposite strategy). We remark that this model, as well as the quantum physical model in general, is epistemic, i.e. it does not represent the internal (and extremely complex) processes in the neural system leading to such transitions. It provides only the formal operator representation of these transitions. However, this formal model is rather rich mathematically; see

[7]It is important to stress that in quantum theory 'interactions' cannot be imagined as actions of forces, as we know them in classical physics. For our applications, the most useful is the information interpretation of quantum theory [46,47]. See Plotnitsky [40] for applications of the information interpretation of quantum theory to DM.

§§3 and 4. We can make the argument that without such an operational reduction of complexity, it would be really impossible to proceed to concrete solutions of dynamical equations. This operational reduction in complexity is one the most important features motivating the exploration of the quantum formalism.

However, the application of this formalism has also deeper foundational consequences. The creation and annihilation operators act on some complex Hilbert space. Therefore, the states of this space can form superpositions. In the quantum formalism, the presence of superpositions of states lead to non-classical probabilistic effects. One of them is the *interference of probabilities.* In quantum physics, this interference effect can be written as a violation of the formula of total probability. The latter is one of the basic laws of CP.

As was mentioned in the introduction, the consideration of superpositions of states is very natural for the modelling of mental phenomena. They represent uncertainty at the level of an individual decision-maker. Moreover, for superpositions, our model exhibits the aforementioned deviation from CP theory, in the form of the violation of the formula of total probability; see §5c.

We also remark that the use of superpositions as initial states generates oscillations (figures 1–3, §4) which are absent for dynamics starting with states representing 'definite strategies', [19].

In short, the use of the quantum formalism for DM can be justified as follows:

— It provides a powerful operational tool for the representation of reflections of decision-makers.

— It provides the unique possibility to represent uncertainty in the mental representation of the decision problem of an *individual decision-maker.*[8]

3. Quantum field theory-inspired model of decision-making in two players game and its dynamics

In this section, we will discuss the details of our (QFT-inspired) model, constructing first the vectors of the players, the Hamiltonian of the system and deducing out of it, the differential equations of motion and their solution, and we will be particularly interested in its asymptotic (in time t) behaviour.

In our game, we have two players, $\mathcal{G}_1$ and $\mathcal{G}_2$. Each player could operate, at $t=0$, two possible choices, 0 and 1. Hence, we have four different possibilities, to which, following Bagarello [19], we associate four different and mutually orthogonal vectors in a four-dimensional Hilbert space $\mathcal{H}_{\mathcal{G}}$. These vectors are $\varphi_{0,0}$, $\varphi_{1,0}$, $\varphi_{0,1}$ and $\varphi_{1,1}$. The first vector, $\varphi_{0,0}$, describes the fact that, at $t=0$, the two players have both chosen 0 ($0_1 0_2$). Of course, such a choice can change during the time evolution of the system. Analogously, $\varphi_{0,1}$ describes the fact that, at $t=0$, the first player has chosen 0, while the second has chosen 1 ($0_1 1_2$). And so on. We see that $\mathcal{F}_\varphi=\{\varphi_{k,l},\ k,l=0,1\}$ is an orthonormal basis for $\mathcal{H}_{\mathcal{G}}$. The general mental state vector of the system $\mathcal{S}_{\mathcal{G}}$ (i.e. of the two players), for $t=0$, is a linear combination

$$\Psi_0=\sum_{k,l=0}^{1}\alpha_{k,l}\varphi_{k,l}, \tag{3.1}$$

where it is natural to assume that $\sum_{k,l=0}^{1}|\alpha_{k,l}|^2=1$ in order to normalize the total probability. Indeed, for instance, we interpret $|\alpha_{0,0}|^2$ as the probability that $\mathcal{S}_{\mathcal{G}}$ is, at $t=0$, in a state $\varphi_{0,0}$, i.e. that both $\mathcal{G}_1$ and $\mathcal{G}_2$ have chosen 0. Analogous interpretations can be given to the other coefficients.

[8]We remark that CP theory also can be used to represent uncertainty. But a probability measure represents uncertainty in an ensemble, i.e. a statistical uncertainty (and not in individual setting). It seems that the quantum representation of uncertainty matches the modelling of uncertainty 'in the head of an individual'; see §5a for further discussion.

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We construct the states $\varphi_{k,l}$ by using two fermionic operators, i.e. two operators b_1 and b_2, satisfying the following *canonical anti-commutation rules* (CAR):

$$\{b_k, b_l^\dagger\} = \delta_{k,l}\mathbb{1} \quad \text{and} \quad \{b_k, b_l\} = 0, \tag{3.2}$$

where $k, l = 0, 1$. Here, $\mathbb{1}$ is the identity operator and $\{x, y\} = xy + yx$ is the anticommutator between x and y. Then, we take $\varphi_{0,0}$ as the vacuum of b_1 and b_2: $b_1\varphi_{0,0} = b_2\varphi_{0,0} = 0$, and build up the other vectors out of it:

$$\varphi_{1,0} = b_1^\dagger\varphi_{0,0}, \quad \varphi_{0,1} = b_2^\dagger\varphi_{0,0} \quad \text{and} \quad \varphi_{1,1} = b_1^\dagger b_2^\dagger\varphi_{0,0}.$$

Remarks 3.1. (1) Note that $\mathcal{F}_\varphi$ could be equivalently constructed starting from the vector $\varphi_{1,1} \in \mathcal{H}$ which is annihilated by $b_1^\dagger$ and $b_2^\dagger$, and then constructing the other vectors of $\mathcal{F}_\varphi$ by acting on $\varphi_{1,1}$ with b_1 and b_2.

(2) The reason why we introduce the operators in (3.2) is because they are used to write down a Hamiltonian-like operator for the system $\mathcal{S}$, from which we deduce the dynamics of the *observables* of $\mathcal{S}$, i.e. the variables needed to describe $\mathcal{S}$, and $\mathcal{S}_\mathcal{G}$ in particular, via the Heisenberg rule.

An explicit representation of these vectors and operators can be found in many textbooks in quantum mechanics [48]. For instance, $\varphi_{k,l} = \varphi_k^{(1)} \otimes \varphi_l^{(2)}$, where $\varphi_0 = \binom{1}{0}$ and $\varphi_1 = \binom{0}{1}$. Then,

$$\varphi_{1,0} = \varphi_1^{(1)} \otimes \varphi_0^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \varphi_{1,1} = \varphi_1^{(1)} \otimes \varphi_1^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and so on. The matrix form of the operators b_j and $b_j^\dagger$ are also quite simple. For instance,

$$b_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad b_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and so on.

Let now $\hat{n}_j = b_j^\dagger b_j$ be the number operator of the jth player: the CAR above implies that $\hat{n}_1\varphi_{k,l} = k\varphi_{k,l}$ and $\hat{n}_2\varphi_{k,l} = l\varphi_{k,l}$, $k, l = 0, 1$. Then, because of what we discussed before, the eigenvalues of these operators correspond to the choice operated by the two players at $t = 0$. For instance, $\varphi_{1,0}$ corresponds to the choice $1_1 0_2$, just because 'one' is the eigenvalue of $\hat{n}_1$ and 'zero' is the eigenvalue of $\hat{n}_2$. It is natural, therefore, to call $\hat{n}_1$ and $\hat{n}_2$ the *strategy operators* (at $t = 0$). Moreover, as b_j and $b_j^\dagger$ modify the attitude of $\mathcal{G}_j$, they can be called the *reflection operators*. We repeat that fermionic operators are in use, because the eigenvalues of $\hat{n}_j$ are exactly 0 and 1, which are the only possible choices of the players of our game.

Our main effort now consists in *giving a dynamics* to the number operators $\hat{n}_j$, following the scheme described in [18]. Therefore, what we first need is to introduce a Hamiltonian H for the system. Then, we will use this Hamiltonian to deduce the dynamics of the number operators as $\hat{n}_j(t) := e^{iHt}\hat{n}_j\, e^{-iHt}$, and finally we will compute the mean values of these operators on some suitable state which describes (see below) the status of the system at $t = 0$. We refer the reader to [18,19] for the details of our construction. Here, we just recall that H is the Hamiltonian of an open system, because the two players $\mathcal{G}_1$ and $\mathcal{G}_2$, in order to take their decision, need also to interact with their environments $\mathcal{R}_1$ and $\mathcal{R}_2$. Contrary to what happens for the players, whose situation can be described in a simple four-dimensional Hilbert space, these environments are naturally defined in an infinite-dimensional Hilbert space. For this reason, they can be thought to describe some subsystem with infinite (or very many) degrees of freedom, as the neurons in the brain, for instance. If we adopt this interpretation, $\mathcal{R}_j$ can be seen as the neural system of $\mathcal{G}_j$, $j = 1, 2$.

The full Hamiltonian H [19], is the following:

$$\left.\begin{aligned} H &= H_0 + H_{\mathrm{I}} + H_{\mathrm{int}},\\ H_0 &= \sum_{j=1}^{2}\omega_j b_j^\dagger b_j + \sum_{j=1}^{2}\int_{\mathbb{R}} \Omega_j(k) B_j^\dagger(k) B_j(k)\,\mathrm{d}k,\\ H_{\mathrm{I}} &= \sum_{j=1}^{2}\lambda_j \int_{\mathbb{R}} (b_j B_j^\dagger(k) + B_j(k) b_j^\dagger)\,\mathrm{d}k\\ \text{and}\quad H_{\mathrm{int}} &= \mu_{\mathrm{ex}}(b_1^\dagger b_2 + b_2^\dagger b_1) + \mu_{\mathrm{coop}}(b_1^\dagger b_2^\dagger + b_2 b_1).\end{aligned}\right\} \tag{3.3}$$

Here, ω_j, λ_j, μ_{ex} and μ_{coop} are real quantities, and $\Omega_j(k)$ are real functions. In analogy with the b_j's, we use fermionic operators $B_j(k)$ and $B_j^\dagger(k)$ to describe the environment:

$$\{B_i(k), B_l(q)^\dagger\} = \delta_{i,l}\delta(k-q)\mathbb{1} \quad \text{and} \quad \{B_i(k), B_j(k)\} = 0, \tag{3.4}$$

which have to be added to those in (3.2). Moreover, each $b_j^\sharp$ anti-commutes with each $B_j^\sharp(k)$: $\{b_j^\sharp, B_l^\sharp(k)\} = 0$ for all j, l and k. Here, $X^\sharp$ stands for X or $X^\dagger$. The various terms of H can be understood as follows: (i) H_0 is the *free* Hamiltonian, which produces no time evolution for the strategy operators $\hat{n}_j$. This is because $[H_0, \hat{n}_j] = 0$, and because H reduces to H_0 in the absence of interactions (i.e. when $\lambda_j = \mu_{\mathrm{ex}} = \mu_{\mathrm{coop}} = 0$). This is in agreement with our idea that the strategies of $\mathcal{G}_1$ and $\mathcal{G}_2$ can be modified only in the presence of interactions. (ii) H_{I} describes the interaction between the players and their neural systems. Of course, the one discussed here is a special kind of interaction, which is useful because it produces an analytical solution for the time evolution of (the mean values of) the strategy operators. Other choices could be considered, but these would, quite likely, break down this nice aspect of the model. (iii) H_{int} describes two different interactions between $\mathcal{G}_1$ and $\mathcal{G}_2$. When $\mu_{\mathrm{coop}} = 0$, the two players act differently, while they behave in the same way when $\mu_{\mathrm{ex}} = 0$. Of course, when both μ_{coop} and μ_{ex} are not zero, the dynamics are even richer. We refer the reader to [19] for more details on (3.3).

The Heisenberg equations of motion $\dot{X}(t) = \mathrm{i}[H, X(t)]$ can now be deduced by using the CAR (3.2) and (3.4) and using H given in (3.3):

$$\left.\begin{aligned} \dot{b}_1(t) &= -\mathrm{i}\omega_1 b_1(t) + \mathrm{i}\lambda_1 \int_{\mathbb{R}} B_1(k,t)\,\mathrm{d}k - \mathrm{i}\mu_{\mathrm{ex}} b_2(t) - \mathrm{i}\mu_{\mathrm{coop}} b_2^\dagger(t),\\ \dot{b}_2(t) &= -\mathrm{i}\omega_2 b_2(t) + \mathrm{i}\lambda_2 \int_{\mathbb{R}} B_2(k,t)\,\mathrm{d}k - \mathrm{i}\mu_{\mathrm{ex}} b_1(t) + \mathrm{i}\mu_{\mathrm{coop}} b_1^\dagger(t)\\ \text{and}\quad \dot{B}_j(k,t) &= -\mathrm{i}\Omega_j(k) B_j(k,t) + \mathrm{i}\lambda_j b_j(t),\end{aligned}\right\} \tag{3.5}$$

$j = 1, 2$. The solution of this system of equations has been found similarly to [19], and it looks like

$$b(t) = \mathrm{e}^{\mathrm{i}Ut} b(0) + \mathrm{i}\int_0^t \mathrm{e}^{\mathrm{i}U(t-t_1)}\beta(t_1)\,\mathrm{d}t_1, \tag{3.6}$$

where we have introduced the following quantities:

$$b(t) = \begin{pmatrix} b_1(t)\\ b_2(t)\\ b_1^\dagger(t)\\ b_2^\dagger(t)\end{pmatrix}, \quad \beta(t) = \begin{pmatrix} \lambda_1\beta_1(t)\\ \lambda_2\beta_2(t)\\ -\lambda_1\beta_1^\dagger(t)\\ -\lambda_2\beta_2^\dagger(t)\end{pmatrix}, \quad U = \begin{pmatrix} \mathrm{i}\nu_1 & -\mu_{\mathrm{ex}} & 0 & -\mu_{\mathrm{coop}}\\ -\mu_{\mathrm{ex}} & \mathrm{i}\nu_2 & \mu_{\mathrm{coop}} & 0\\ 0 & \mu_{\mathrm{coop}} & \mathrm{i}\bar{\nu}_1 & \mu_{\mathrm{ex}}\\ -\mu_{\mathrm{coop}} & 0 & \mu_{\mathrm{ex}} & \mathrm{i}\bar{\nu}_2\end{pmatrix}$$

and where $\Omega_j(k) = \Omega_j k$, $\Omega_j > 0$, $\nu_j = \mathrm{i}\omega_j + \pi(\lambda_j^2/\Omega_j)$ and $\beta_j(t) = \int_{\mathbb{R}} B_j(k)\,\mathrm{e}^{-\mathrm{i}\Omega_j k t}\,\mathrm{d}k$, $j = 1, 2$.

As already stated, the next step consists in taking the average of the time evolution of the strategy operators, $\hat{n}_j(t) = b_j^\dagger(t) b_j(t)$, on a state over the full system $\mathcal{S} = \mathcal{S}_{\mathcal{G}} \otimes \mathcal{R}$, where $\mathcal{S}_{\mathcal{G}}$ has already been introduced, $\mathcal{S}_{\mathcal{G}} = \{\mathcal{G}_1, \mathcal{G}_2\}$, and $\mathcal{R} = \{\mathcal{R}_1, \mathcal{R}_2\}$. These states are assumed to be tensor products of vector states for $\mathcal{S}_{\mathcal{G}}$ and states on the environment in the following way: for each

operator of the form $X_{\mathcal{S}} \otimes Y_{\mathcal{R}}$, $X_{\mathcal{S}}$ being an operator of $\mathcal{S}_{\mathcal{G}}$ and $Y_{\mathcal{R}}$ an operator of the environment, we have

$$\langle X_{\mathcal{S}} \otimes Y_{\mathcal{R}}\rangle := \langle \Psi_0, X_{\mathcal{S}}\Psi_0\rangle \omega_{\mathcal{R}}(Y_{\mathcal{R}}).$$

Here, Ψ_0 is the vector introduced in (3.1), while $\omega_{\mathcal{R}}(.)$ is a state satisfying the following standard properties [49]:

$$\omega_{\mathcal{R}}(\mathbb{1}_{\mathcal{R}}) = 1, \quad \omega_{\mathcal{R}}(B_j(k)) = \omega_{\mathcal{R}}(B_j^\dagger(k)) = 0 \quad \text{and} \quad \omega_{\mathcal{R}}(B_j^\dagger(k)B_l(q)) = N_j\delta_{j,l}\delta(k-q), \tag{3.7}$$

for some constant N_j. Also, $\omega_{\mathcal{R}}(B_j(k)B_l(q)) = 0$ for all j and l.

Remark 3.2. At first sight, this expression for the state on $\mathcal{S}$ introduces a sort of asymmetry between $\mathcal{S}_{\mathcal{G}}$ and $\mathcal{R}$, because their states look to be of a different nature: the one over $\mathcal{S}_{\mathcal{G}}$ is a vector state, while the one over $\mathcal{R}$ is a linear positive functional over the algebra of the fermionic operators $B_j(k)$ and $B_j^\dagger(k)$. This is not really surprising, because $\mathcal{S}_{\mathcal{G}}$ involves just two degrees of freedom, while $\mathcal{R}$ involves an infinite number of degrees of freedom. Nevertheless, using the so-called Gelfand–Naimark–Segal construction [45], it is possible to present $\omega_{\mathcal{R}}$ as a vector state.

After a few computations, noting $V(t) = e^{iUt}$ and calling $V_{k,l}(t)$ its (k,l)-matrix element, we deduce the following general formulae for the DFs of $\mathcal{G}_1$ and $\mathcal{G}_2$, which extend those found in [19] in the absence of interference:

$$\left.\begin{aligned} & n_1(t) = \langle (b_1^\dagger(t)b_1(t))^\dagger\rangle = \mu_1^{(\mathcal{G})}(t) + \delta\mu_1^{(\mathcal{G})}(t) + n_1^{(B)}(t) \\ \text{and} \quad & n_2(t) = \langle (b_2^\dagger(t)b_2(t))^\dagger\rangle = \mu_2^{(\mathcal{G})}(t) + \delta\mu_2^{(\mathcal{G})}(t) + n_2^{(B)}(t). \end{aligned}\right\} \tag{3.8}$$

Here, we have introduced

$$\left.\begin{aligned} & \mu_1^{(\mathcal{G})}(t) = |V_{1,1}(t)|^2(|\alpha_{1,0}|^2 + |\alpha_{1,1}|^2) + |V_{1,2}(t)|^2(|\alpha_{0,1}|^2 + |\alpha_{1,1}|^2) \\ & \qquad + |V_{1,3}(t)|^2(|\alpha_{0,0}|^2 + |\alpha_{0,1}|^2) + |V_{1,4}(t)|^2(|\alpha_{0,0}|^2 + |\alpha_{1,0}|^2) \\ \text{and} \quad & \mu_2^{(\mathcal{G})}(t) = |V_{2,1}(t)|^2(|\alpha_{1,0}|^2 + |\alpha_{1,1}|^2) + |V_{2,2}(t)|^2(|\alpha_{0,1}|^2 + |\alpha_{1,1}|^2) \\ & \qquad + |V_{2,3}(t)|^2(|\alpha_{0,0}|^2 + |\alpha_{0,1}|^2) + |V_{2,4}(t)|^2(|\alpha_{0,0}|^2 + |\alpha_{1,0}|^2). \end{aligned}\right\} \tag{3.9}$$

$$\left.\begin{aligned} & \delta\mu_1^{(\mathcal{G})}(t) = 2\Re[\overline{V_{1,1}(t)}V_{1,2}(t)\bar{\alpha}_{1,0}\alpha_{0,1} + \overline{V_{1,1}(t)}V_{1,4}(t)\bar{\alpha}_{1,1}\alpha_{0,0}] \\ & \qquad - 2\Re[\overline{V_{1,2}(t)}V_{1,3}(t)\bar{\alpha}_{1,1}\alpha_{0,0} + \overline{V_{1,3}(t)}V_{1,4}(t)\bar{\alpha}_{0,1}\alpha_{1,0}] \\ \text{and} \quad & \delta\mu_2^{(\mathcal{G})}(t) = 2\Re[\overline{V_{2,1}(t)}V_{2,2}(t)\bar{\alpha}_{1,0}\alpha_{0,1} + \overline{V_{2,1}(t)}V_{2,4}(t)\bar{\alpha}_{1,1}\alpha_{0,0}] + \\ & \qquad - 2\Re[\overline{V_{2,2}(t)}V_{2,3}(t)\bar{\alpha}_{1,1}\alpha_{0,0} + \overline{V_{2,3}(t)}V_{2,4}(t)\bar{\alpha}_{0,1}\alpha_{1,0}] \end{aligned}\right\} \tag{3.10}$$

$$\left.\begin{aligned} & n_1^{(B)}(t) = 2\pi\int_0^t \mathrm{d}t_1 \left[\frac{\lambda_1^2}{\Omega_1}(|V_{1,1}(t-t_1)|^2 N_1 + |V_{1,3}(t-t_1)|^2(1-N_1))\right] \\ & \qquad + 2\pi\int_0^t \mathrm{d}t_1 \left[\frac{\lambda_2^2}{\Omega_2}(|V_{1,2}(t-t_1)|^2 N_2 + |V_{1,4}(t-t_1)|^2(1-N_4))\right] \\ \text{and} \quad & n_2^{(B)}(t) = 2\pi\int_0^t \mathrm{d}t_1 \left[\frac{\lambda_1^2}{\Omega_1}(|V_{2,1}(t-t_1)|^2 N_1 + |V_{2,3}(t-t_1)|^2(1-N_1))\right] \\ \text{and} \quad & \qquad + 2\pi\int_0^t \mathrm{d}t_1 \left[\frac{\lambda_2^2}{\Omega_2}(|V_{2,2}(t-t_1)|^2 N_2 + |V_{2,4}(t-t_1)|^2(1-N_4))\right]. \end{aligned}\right\} \tag{3.11}$$

In formula (3.8), we have clearly divided contributions of three different natures: $\mu_j^{(\mathcal{G})}(t)$ contains contributions only due to the players $\mathcal{G}_1$ and $\mathcal{G}_2$. Their analytic expressions become particularly simple if the initial state Ψ_0 is just one of the vectors $\varphi_{j,k}$, i.e. if all the coefficients $\alpha_{k,l}$ in (3.1) are zero, except one. In this particular situation, all the contributions in $\delta\mu_j^{(\mathcal{G})}(t)$, which again only

refer to $\mathcal{G}_1$ and $\mathcal{G}_2$, are zero. For this reason, we call $\delta\mu_1^{(\mathcal{G})}(t)$ and $\delta\mu_2^{(\mathcal{G})}(t)$ *interference terms*: they are only present if Ψ_0 is some superposition of eigenvectors of the (time zero) number operators. Otherwise, they simply disappear. Finally, $n_1^{(B)}(t)$ and $n_2^{(B)}(t)$ arise because of the interaction of the players with the environments: as we see from (3.11), they are both zero if λ_1 and λ_2 in the Hamiltonian are both equal to zero, and they do not depend on the explicit form of Ψ_0.

A detailed analysis of what happens when there are no interference terms can be found in [19], where the focus was mainly on the asymptotic behaviour of the DFs in the absence of interferences. Here, on the other hand, we want to analyse what happens when the interference does exist already at $t=0$ (i.e. when more than just one coefficient $\alpha_{j,k}$ is non-zero), and we are also interested in the behaviour of the DFs for finite time.

4. Analysis of the results

In the figures plotted in this section, we will call $\mathcal{C}_1$ the following choice of parameters of H: $\omega_1=1$, $\omega_2=2$, $\Omega_1=\Omega_2=0.1$, $\lambda_1=\lambda_2=0.5$, and $\mathcal{C}_2$ the second choice: $\omega_1=0.1$, $\omega_2=0.2$, $\Omega_1=\Omega_2=1$, $\lambda_1=1$ and $\lambda_2=0.7$. Also, we call $\mathcal{C}_{\alpha,1}$ and $\mathcal{C}_{\alpha,2}$ the following choices of the parameters $\alpha_{k,l}$ in (3.1): $\mathcal{C}_{\alpha,1}=\{\alpha_{k,l}=\frac{1}{2}, \forall\, k,l\}$, while $\mathcal{C}_{\alpha,2}=\left\{\alpha_{0,1}=\frac{1}{2}=-\alpha_{1,1}, \alpha_{0,0}=\frac{i}{2}=-\alpha_{1,0}\right\}$.

Of course, several other choices could also be considered. However, the ones we are fixing here cover already different situations. In particular, while in $\mathcal{C}_1$ the interaction parameters λ_1 and λ_2 are equal, they are different in $\mathcal{C}_2$. Also, while in $\mathcal{C}_1$ each ω_j is bigger than each Ω_k, the opposite holds for $\mathcal{C}_2$. This is important, because it is known that the ω_j's are related to the inertia of the player $\mathcal{G}_j$ [18,19]. Moreover, as for the choices $\mathcal{C}_{\alpha,1}$ and $\mathcal{C}_{\alpha,2}$, the difference is clear: in $\mathcal{C}_{\alpha,1}$ all the coefficients in (3.1) are equal, and in particular there is no relative phase between the various $\varphi_{k,l}$'s in Ψ_0, whilst this is not so when adopting the choice $\mathcal{C}_{\alpha,2}$. And, as we will see, this makes indeed a big difference: we clearly see that, even if (as expected from what is deduced in [19]) the asymptotic values of the two DFs appear to be independent of the choice of $\mathcal{C}_{\alpha,1}$ and $\mathcal{C}_{\alpha,2}$, there exists a certain time window in which the choice of the $\alpha_{k,l}$'s really change the behaviours of the functions. More concretely, if we add a phase in the coefficients defining the original vector Ψ_0, we may observe quite large oscillations. Then, *interference terms in Ψ_0 make it, in general, quite difficult to get a decision*. In particular, this is the effect of the relative phases in the interference coefficients (figure 1*b*,*d*), while if these coefficients have all the same phases, a decision can be reached quite soon (figure 1*a*,*c*). However, if we wait for a sufficiently long time, in both cases we reach the same final values of the DFs: the asymptotic values of the DFs only depend on the state of the environment, and not on the particular choice of Ψ_0.

These conclusions are confirmed by other choices of the state on the environment. For instance, we plot again the DFs for the choices $\mathcal{C}_{\alpha,1}$ (figure 2*a*,*c*) and $\mathcal{C}_{\alpha,2}$ (figure 2*b*,*d*) and for the same choice of the parameters as in figure 1, while the state of the environment is chosen different, because we take now $N_1=N_2=1$. We see that there is no particular difference between the two DFs $n_1(t)$ and $n_2(t)$ (this is possibly due to the fact that $N_1=N_2$). However, adding the phases to the $\alpha_{k,l}$ creates, again, a lot of noise in the DM process, noise which disappears, but only after a sufficiently long time.

It is useful to stress that, changing further the parameters of the Hamiltonian, does not really affect our conclusions. Indeed, figure 3 shows that even with different choices of the parameters in H, the choice $\mathcal{C}_2$ with $\mu_{\mathrm{ex}}=100$ (rather than $\mu_{\mathrm{ex}}=500$, as in the previous figures), phases create noise, and again, this noise becomes smaller and smaller after some time.

In this case, the existence of an asymptotic limit for the DFs is less evident, but this is only due to the *small* time interval considered. It is not hard to imagine that we could still recover a clear asymptotic value for $n_j(t)$ if we consider a time interval larger than just $[0,0.5]$. However, we will not do this here in order to keep the range of the figures uniform.

So far, we have taken $\mu_{\mathrm{coop}}=0$. The same behaviour is observed if we take $\mu_{\mathrm{coop}}\neq 0$: we could check (but we do not include the plots here) that the choice $\mathcal{C}_{\alpha,2}$ is again much more noisy than $\mathcal{C}_{\alpha,1}$.

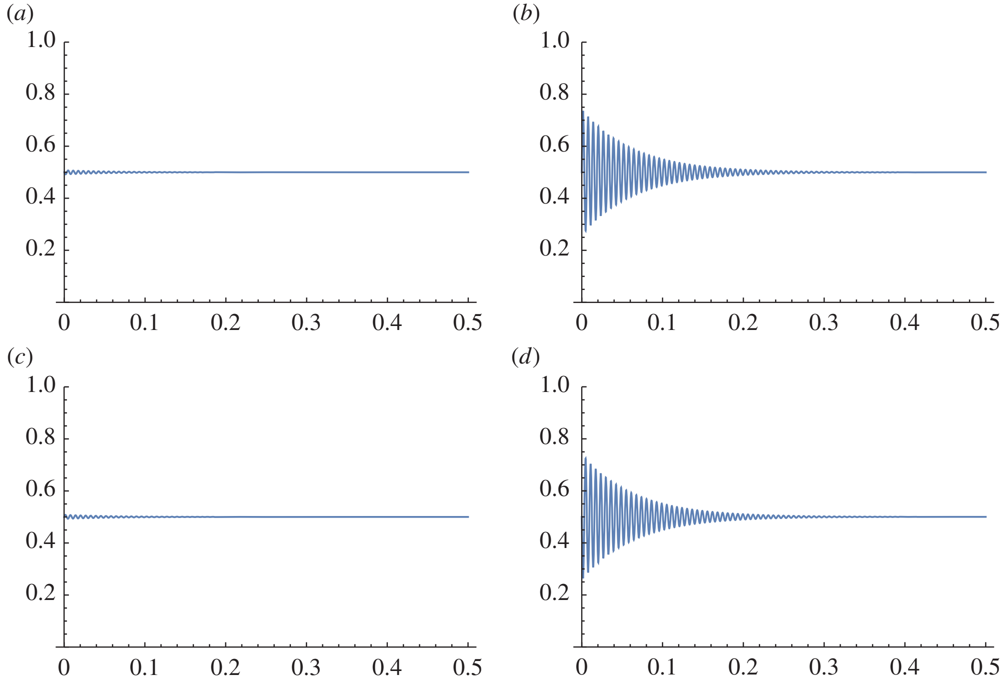

Figure 1. The DFs $n_1(t)$ (*a*,*b*) and $n_2(t)$ (*c*,*d*) for parameters $\mathcal{C}_1$, $N_1 = 0$, $N_2 = 1$, $\mu_{\rm ex} = 500$, $\mu_{\rm coop} = 0$ and for $\mathcal{C}_{\alpha,1}$ (*a*,*c*) and $\mathcal{C}_{\alpha,2}$ (*b*,*d*). (Online version in colour.)

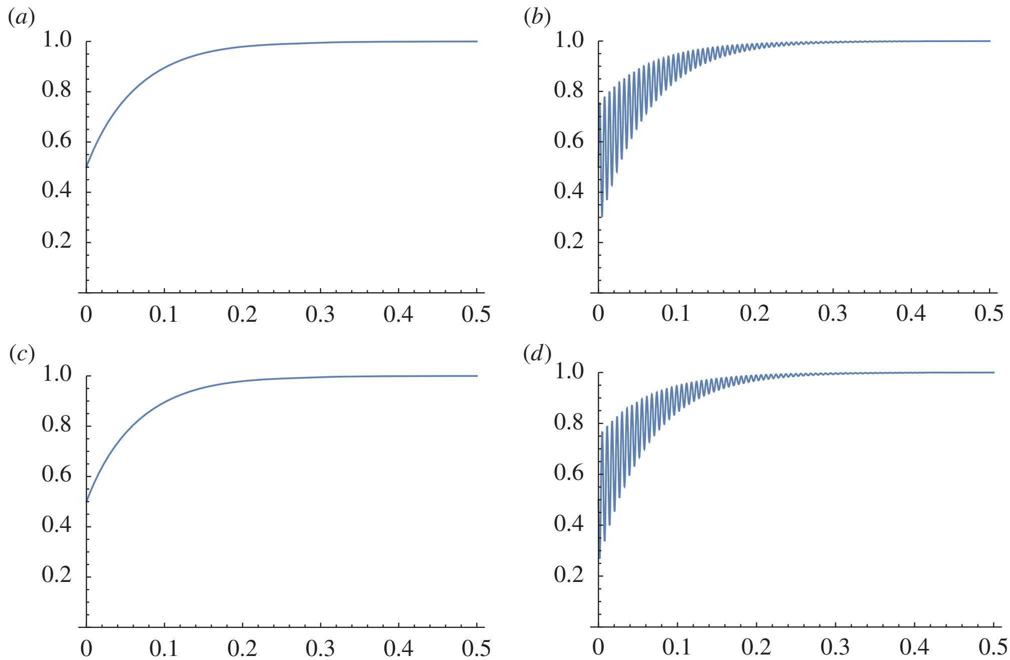

Figure 2. The DFs $n_1(t)$ (*a*,*b*) and $n_2(t)$ (*c*,*d*) for parameters $\mathcal{C}_1$, $N_1 = 1$, $N_2 = 1$, $\mu_{\rm ex} = 500$, $\mu_{\rm coop} = 0$ and for $\mathcal{C}_{\alpha,1}$ (*a*,*c*) and $\mathcal{C}_{\alpha,2}$ (*b*,*d*). (Online version in colour.)

Also, no particular difference arises if we consider both $\mu_{\rm ex}$ and $\mu_{\rm coop}$ different from zero, especially when they are different enough (again, we do not include the plots here). However, interestingly enough, this noise is not so evident if $\mu_{\rm ex}$ and $\mu_{\rm coop}$ are not so different (figure 4). It

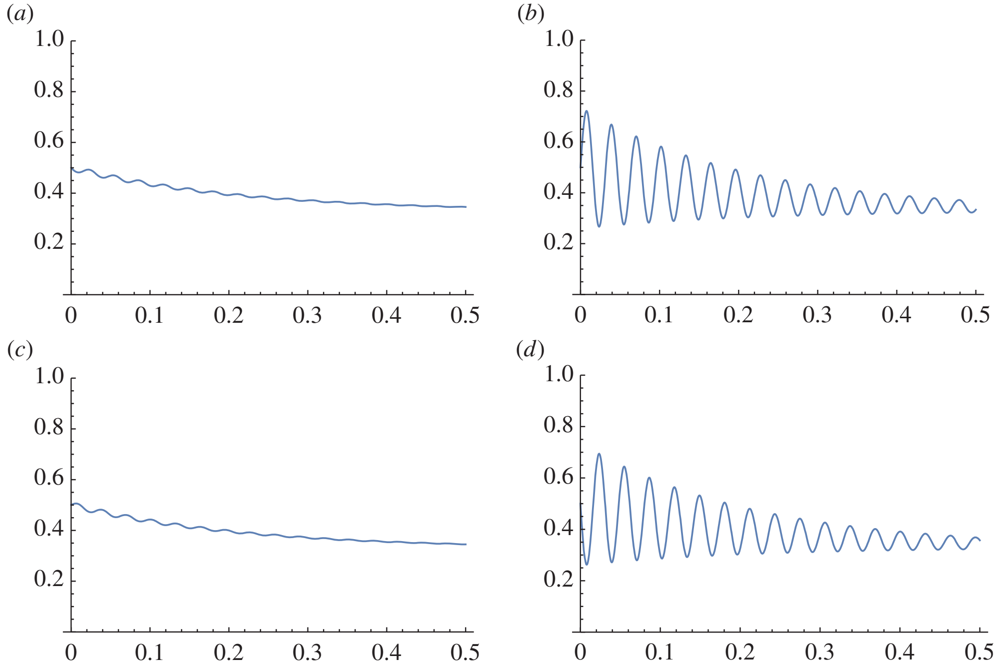

Figure 3. The DFs $n_1(t)$ (*a*,*b*) and $n_2(t)$ (*c*,*d*) for parameters $\mathcal{C}_2$, $N_1 = 0$, $N_2 = 1$, $\mu_{\rm ex} = 100$, $\mu_{\rm coop} = 0$ and for $\mathcal{C}_{\alpha,1}$ (*a*,*c*) and $\mathcal{C}_{\alpha,2}$ (*b*,*d*). (Online version in colour.)

seems that when the two terms in $H_{\rm int}$ (see (3.3)) act in cooperation, they become capable to *filter the noise*, making the effect of the phases in $\alpha_{k,l}$ not so strong. This is an interesting feature, which is surely worthy of a deeper analysis.

5. Discussion

In this section, we shall discuss the output of our model in detail.

(a) State's interpretation

As is well known, the present situation in quantum foundations is characterized by a huge diversity of interpretations of the quantum state. The two main classes of interpretations correspond to *the statistical and individual viewpoints on a quantum state* [50] for details. By the former a quantum state encodes probabilities for the results of measurements for an ensemble of identically prepared quantum systems.

We investigated the behaviour of the averages of the strategy operators of the two players $\mathcal{G}_1$ and $\mathcal{G}_2$ interacting with the corresponding environments (mental baths) $\mathcal{R}_1$ and $\mathcal{R}_2$. One of the main outputs of our quantum-like model is that these averages stabilize for $t \to \infty$ (for natural Hamiltonians describing the interaction between players and their interactions with mental baths). To interpret this result, we have to fix one of the interpretations of a quantum state.

The straightforward interpretation of this stabilization result can be presented on the basis of the statistical interpretation of quantum states. Here, we consider a very large ensemble of pairs of players, and the averages of the strategy operators are treated as *the ensemble averages.* Our stabilization result implies that, on average (with respect to this ensemble), the strategies of players stabilize to fixed values in the very long run of the DM reflections (this is encoded in creation and annihilation operators forming the Hamiltonian of the game).

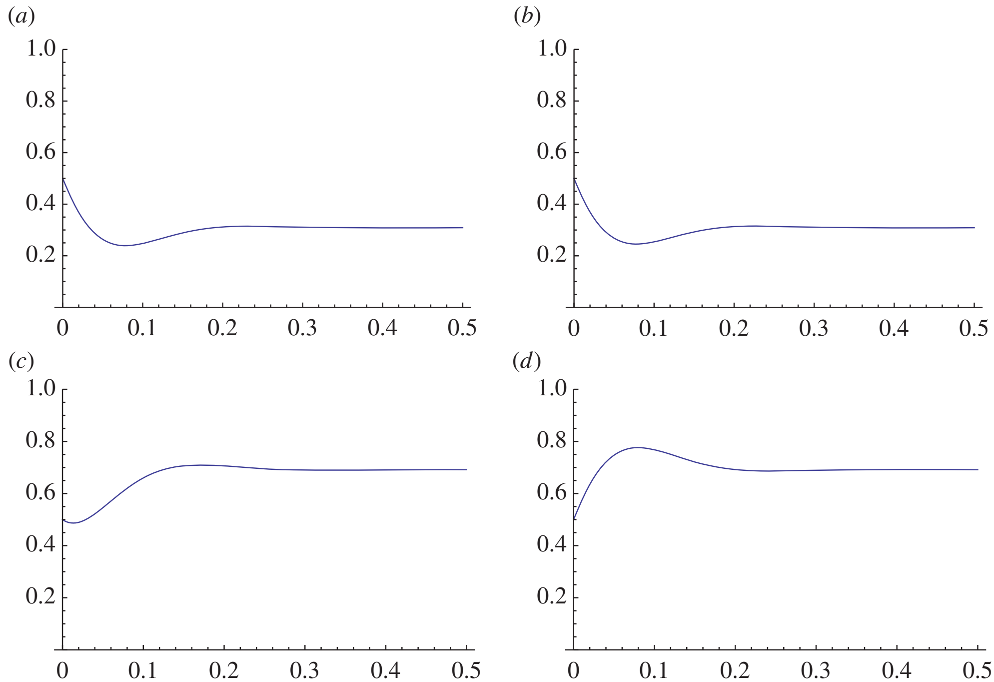

Figure 4. The DFs $n_1(t)$ (*a*,*b*) and $n_2(t)$ (*c*,*d*) for parameters $\mathcal{C}_1$, $N_1 = 0$, $N_2 = 1$, $\mu_{\mathrm{ex}} = 10$, $\mu_{\mathrm{coop}} = 10$ and for $\mathcal{C}_{\alpha,1}$ (*a*,*c*) and $\mathcal{C}_{\alpha,2}$ (*b*,*d*). (Online version in colour.)

Now, we switch to the individual interpretation of a quantum state. Here, the question of the interpretation of probabilities encoded in a quantum state is more complicated than in the case of the statistical interpretation. It seems that, for our applications of the mathematical formalism of quantum mechanics to the operational modelling of DM, the most natural interpretation of probability is *the subjective interpretation*. Recently, this interpretation became popular in quantum information theory and it is known as Quantum Bayesianism (QBism) [51,52].[9]

In the QBist framework, the average stabilization output of our model can be interpreted in the following way:

At the initial instant of time $t = 0$, each pair of players assigns their individual subjective probabilities to selections of their strategies,

$$p_i(j) \equiv p_i(j;0), \quad i = 1,2,\ j = 0,1.$$

These probabilities depend on the initial state Ψ_0 of the pair of players. By using the representation of Ψ_0 as superposition of the 'certainty states' $\varphi_{k,l}$ (see (3.1)) and Born's rule, these probabilities can be represented as

$$p_1(j) = |\alpha_{j,0}|^2 + |\alpha_{j,1}|^2 \quad \text{and} \quad p_2(j) = |\alpha_{0,j}|^2 + |\alpha_{1,j}|^2,$$

for $j = 0,1$. In fact, for instance, we have $p_1(0) = |\langle\varphi_{00}, \Psi_0\rangle|^2 + |\langle\varphi_{01}, \Psi_0\rangle|^2$ and so on. If Ψ_0 is non-factorizable (entangled), then $p_1(j)$ cannot be expressed solely in terms of the state of $\mathcal{G}_1$ and vice

[9]We do not claim that QBism is the proper interpretation of quantum *physics*. See Khrennikov [50] for reflections from one of the authors of this paper on the applicability of QBism in quantum physics. But it is very natural to use it for our purpose [33,50] for a motivation. We also remark that originally classical game theory and the theory of DM was formulated by von Neumann & Morgenstern [7] in the statistical probabilistic framework: probabilities were treated from the frequentist viewpoint. However, Savage [53] reformulated it by using the subjectivist viewpoint on probability. Nowadays, the latter is dominating the foundations of DM and the axiomatic foundation of economics, but with the strong emphasize of the role of Bayesian inference.

versa. We emphasize that, in the case of entanglement, for a pure state Ψ_0, the states of players are mixed states and they are represented by density operators.

Starting with the initial state Ψ_0, and hence with the probabilities $p_i(j)$, each decision maker updates continuously her state by taking into account the impact of the environment and the feedback from the update of the state of the co-player. This state dynamics studied in previous sections generates the corresponding dynamics of the subjective probabilities, $t \to p_i(j;t)$. The essence of exploring the quantum-like model indicates that the genuine dynamics are not a dynamics of probabilities, but rather a dynamics of the state. Metaphorically, we can say that the dynamics of probabilities are a shadow of the mental state dynamics. The state dynamics can be expressed (see §3) in terms of the strategy operators.

A direct computation shows that the strategy operators $\hat{n}_i$ (at $t=0$) can be represented in the form (spectral decomposition)

$$\hat{n}_1 = |\varphi_{1,0}\rangle\langle\varphi_{1,0}| + |\varphi_{1,1}\rangle\langle\varphi_{1,1}| \tag{5.1}$$

and

$$\hat{n}_2 = |\varphi_{0,1}\rangle\langle\varphi_{0,1}| + |\varphi_{1,1}\rangle\langle\varphi_{1,1}|, \tag{5.2}$$

where $(|f\rangle\langle g|)h = \langle g,h\rangle f$ for all $f,g,h \in \mathcal{H}$. The ones in (5.1)–(5.2) are orthogonal projectors onto the subspaces L_1 with the basis $(\varphi_{1,0},\varphi_{1,1})$ and L_2 with the basis $(\varphi_{0,1},\varphi_{11})$. Those are the specifics of the fermionic representation of the reflection and strategy operators.

Now, with simple manipulations, we observe that

$$\begin{aligned} p_1(1;t) &= |\langle\varphi_{10},\Psi(t)\rangle|^2 + |\langle\varphi_{11},\Psi(t)\rangle|^2 = \langle\Psi(t),(|\varphi_{1,0}\rangle\langle\varphi_{1,0}| + |\varphi_{1,1}\rangle\langle\varphi_{1,1}|)\Psi(t)\rangle \\ &= \langle\Psi(t),\hat{n}_1\Psi(t)\rangle = \langle\Psi_0,\hat{n}_1(t)\Psi_0\rangle, \end{aligned}$$

moving from the Schrödinger to the Heisenberg representation. Therefore, the averages of these operators, $i=1,2$, are equal to the probabilities of selecting the decision $s=1$:

$$p_i(1;t) = \langle\hat{n}_i(t)\rangle, \quad i=1,2. \tag{5.3}$$

These are nothing else than DFs considered in §3 (see (3.8)). This clarifies the relation between the subjective probabilities and the DFs.

(b) The process of decision-making: the subjective probability viewpoint

Each decision-maker starts with assignment of the subjective probabilities $p_i(1) = \langle\hat{n}_i\rangle, p_i(0) = 1 - \langle\hat{n}_i\rangle$, where $\hat{n}_i = \hat{n}_i(0)$. Then, she begins her reflections on the possible selections of the strategies (adapted to the environment and interactions with the other decision-maker) and her subjective probabilities fluctuate; see §§3 and 4.

We have seen that the structure of fluctuations depends crucially on the initial state Ψ_0. In the process of DM, the magnitude of these fluctuations decreases and subjective probabilities stabilize to the fixed values. These are the probabilities which are used by the decision-maker to make her choice. After the determination of her subjective probabilities, she proceeds as a classical decision-maker. For the instant of time t, the odds in favour of the decision labelled by 1 are given by the proportion

$$O_i(1;t) = \frac{p_i(1;t)}{p_i(0;t)}. \tag{5.4}$$

We consider the instance of time τ when the decision is made. This τ corresponds to a diminishing of the fluctuations to the minimal level. In the theoretical model, $\tau = \infty$. However, in reality a decision-maker cannot wait for an infinite time to make the decision. The decision-maker uses some threshold for fluctuations $\epsilon > 0$ and the decision instant τ is determined by ϵ.

If $O_i(1;\tau) > 1$, then $\mathcal{G}_i$ makes the decision labelled by 1. If $O_i(1;\tau) < 1$, then she makes the opposite decision (if $O_i(1;\tau) = 1$, she either makes the decision randomly with probability $\frac{1}{2}$ or she repeats the process of DM with a modified initial state Ψ_0). We remark that, in the absence of the

mental baths, the subjective probabilities assigned by a decision-maker to the possible strategies would fluctuate forever, as deduced, for instance, in [20] in a different context.

(c) Violation of the law of total probability: 'interference of probabilities'

As was emphasized by Feynman & Hibbs [54], the probabilistic data from interference experiments with quantum systems, e.g. the two-slit experiment, can be treated as violating the laws of CP. They pointed to a violation of the additivity of probability. Feynman's argument was reformulated by one of the coauthors of this paper in terms of conditional probabilities—as a violation of the law of total probability (LTP) [35]. In this form, this argument is nicely applicable to the process of DM. The classical Bayesian scheme of DM is fundamentally based on the LTP. Its violation leads to non-Bayesian schemes of DM and the quantum formalism provides one of such schemes.[10]

We recall the LTP. Consider two random variables ξ and η taking discrete values, $\xi = \alpha_1, \ldots, \alpha_N$ and $\eta = \beta_1, \ldots, \beta_M$. Then, in the classical probabilistic framework (the measure-theoretic model of probability [55] it can be proved that

$$p(\eta = \beta) = \sum_{\alpha} p(\xi = \alpha) p(\eta = \beta | \xi = \alpha). \tag{5.5}$$

It is important to remark that the conditional probability is defined by the Bayes formula: $p(\eta = \beta \,|\, \xi = \alpha) = p(\eta = \beta, \xi = \alpha)/p(\xi = \alpha)$, where $p(\eta = \beta, \xi = \alpha)$ is the joint probability distribution of the pair of random variables. As is well known, in general, for quantum observables, the joint probability is not well defined. Therefore, the classical Bayes formula for conditional probabilities is inapplicable and one can expect that, for quantum conditional probabilities which are defined in the Hilbert space formalism, the formula (5.5) can be violated [35].

The classical LTP in our setting can be formulated as follows. We remind that the probabilities $p_i(s;t), s = 0, 1$, depend on the initial state Ψ_0, i.e. in fact, we could even use the notation $p_i(s;t|\Psi_0)$. Among the possible initial mental states of the pairs of players $(\mathcal{G}_1, \mathcal{G}_2)$, the states $\varphi_{k,m}, k, m = 0, 1$, play a very special role. Here, initially the players were completely sure in their strategies, $\mathcal{G}_1$ with the strategy k and $\mathcal{G}_2$ with the strategy m. In (5.5), the probabilities $p_i(s;t|k,m) \equiv p_i(s;t|\varphi_{k,m}), s = 0, 1$, can be identified with the conditional probabilities $p(\eta = \beta | \xi = \alpha)$ with $\alpha = (k, m)$ and $\beta = 0, 1$. The probabilities

$$p_i(k, m) = |\langle \Psi_0 \,|\, \varphi_{k,m} \rangle|^2, \quad k, m = 0, 1, \tag{5.6}$$

encode uncertainty in the determination of the pairs of strategies (k, m) for the initial mental state Ψ_0. In (5.5), they are identified with the probabilities $p(\xi = \alpha)$.

The interference-like representation of the DFs (3.8) can be rewritten as a quantum generalization of the LTP:

$$\begin{aligned} p_i(s;t) = &\sum_{(k,m)} p_i(k,m) p_i(s;t \,|\, k,m) \\ &+ \sum_{(k,m) \neq (j,n)} \cos\theta_i(s;t \,|\, (k,m,j,n) \sqrt{p_i(k,m) p_i(s;t \,|\, k,m) p_i(j,n) p_i(\alpha;t \,|\, j,n)}. \end{aligned} \tag{5.7}$$

Here, the phase $\theta_i(s;t \,|\, (k,m,j,n)$ encodes a sort of interference between the pairs of strategies (k,m) and (j,n). In our quantum model, this phase can be easily calculated by using the phases of the states Ψ_0 and $\varphi_{k,m}, \varphi_{j,n}$.

As was shown in [35], in the purely probabilistic framework (i.e. without direct appealing to the Hilbert space formalism) a violation of the LTP means the impossibility to embed all probabilities in (5.7), into a single CP space. Thus, the appearance of the additional interference-like term disturbing the classical LTP (see (5.7)) shows that CP has the restricted domain of applications. For our applications to DM, it is more useful to interpret this result in terms of the

[10]Of course, a non-Bayesian scheme need not be precisely quantum. It is not clear whether the quantum formalism can cover all probabilistic phenomena arising in DM [22,23].

interrelation between classical and non-classical logics. CP theory is based on classical Boolean logic. Therefore, the impossibility of using CP implies the impossibility of using the Boolean logic. Thus, in our quantum-like model the actions of the players are not ruled solely by the laws of Boolean logics. They can make decisions with reasoning based on non-classical logic.

6. Conclusion and perspectives

The DM under uncertainty plays a crucial role in economics, game theory and many areas of social science. Until now, the most intensive and successful modelling of the DM process was performed on the basis of Bernoulli's idea to explore utility functions (which originated in his attempting to solve the St Petersburg paradox). Several theories have ensued since then: expected utility theory, prospect theory, cumulative prospect theory and other approaches. In spite of the aforementioned success of these utility-based models, it is impossible to ignore the growing dissatisfaction by the present state of the art in DM. In particular, the number of paradoxes have increased over the years: Allais, Ellsberg, Machina, and one recently counted 39 paradoxes [56]. Already, in subjective uncertainty utility models, the representation of the agent's utility by a function is a fuzzy problem. Individuals may use a huge variety of possible utility functions and the determination of the class of possible functions is a complicated problem [57–59].

One can treat DM as the interaction of an agent with a complex information environment which includes a variety of behavioural, economic, social and geopolitical factors as well as beliefs about states of other agents (e.g. the financial market). Therefore, it is natural to model DM as an (environment-)adaptive dynamical process. In principle, one may try to encode such a complex information environment by a single function encoding the utility of a context. However, this would definitely be a high simplification of the mathematical representation of the DM context.

The most advanced mathematical model of interaction of a system with an environment is presented in quantum theory. The modern quantum information viewpoint on quantum theory [40,46,47] justifies the possibility to apply this formalism outside of physics. Here, the keyword is *adaptivity* to the environment. In this paper, we have applied the quantum model of adaptive dynamics to the modelling of the DM process. Our model is QFT-inspired: the information environment is modelled with the aid of quantum fields which are treated as operational quantities carrying information. In such a DM model, an agent does not maximize (expected) utility, but she searches for decisions matching 'demands' of the environment and (agent's representation) of the belief states of other agents involved in the DM process. This dynamical adaptive process is based on the representation of the DM context by a quantum pure state carrying maximal available information about the situation.

We have found analytical and numerical solutions for operator-valued functions representing the DM process and their expectations, and we considered the problem of stabilization for $t \to \infty$. The output of stabilization is considered as the classical (mixed) decision strategy. Of course, the real DM process cannot take an infinite time and we are interested in approximate stabilization. We have carefully analysed the interpretational issues of our model (see §5). In particular, we have looked at the coupling between subjective probability and the private agent interpretation of quantum mechanics (QBism) [51,52].

This paper is of a conceptual nature. Its aim is to present a quite general model of DM under uncertainty as an adaptive dynamical process of evolution of the belief state of an agent interacting with a complex information environment (e.g. the financial market). This paper also wishes to demonstrate the mathematical power of this general model. Our proposed model can be explored for a variety of problems in economics, sociology and politics. See Bagarello [19–21] and Khrennikova [38][11] for concrete applications of special variations of the presented model. Others are in progress.

Data accessibility. This article has no additional data.

[11] We remark that the QFT-inspired model generalizes the quantum dynamical DM-model of Pothos & Busemeyer [42]. It also has close connections with the dynamical DM-model based on the quantum master equation [14–17].

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Authors' contributions. F.B. proposed the model and its solution. A.K. and E.H. worked on the interpretation of the model in the context of DM.

Competing interests. We declare we have no competing interests.

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References

1. Machina MJ. 2003 States of the world and the state of decision theory. In *The economics of risk* (ed. D Meyer). Kalamazoo, MI: W.E. Upjohn Institute for Employment Research.
2. Detemple J, Rindisbacher M. 2013 A structural model of dynamic market timing. *Rev. Financ. Stud.* **16**, 2492–47. (doi:10.1093/rfs/hht028)
3. Detemple J, Rindisbacher M. 2016 The private information price of risk. In *Handbook of post-crisis financial modelling* (eds E Haven, P Molyneux, JOS Wilson, S Fedotov, M Duygun), pp. 190–213. London, UK: Palgrave-MacMillan Publishers.
4. Bikhchandani S, Hirshleifer J, Riley JG. 2013 *The analytics of uncertainty and information*. Cambridge, UK: Cambridge University Press.
5. Björk T. 1998 *Arbitrage theory in continuous time*. Oxford, UK: Oxford University Press.
6. Ma C. 2011 *Advanced asset pricing theory*. London, UK: Imperial College Press.
7. von Neumann J, Morgenstern O. 1944 *Theory of games and economic behavior*. Princeton, NJ: Princeton University Press.
8. Allais M. 1953 Le comportement de l'homme rationnel devant le risque: critique des postulats et axiomes de l'école amé. *Econometrica*, **21**, 503–546. (doi:10.2307/1907921)
9. Ellsberg D. 1961 Risk, ambiguity, and the Savage axioms. *Q. J. Econ.* **75**, 643–669. (doi:10.2307/1884324)
10. Machina MJ. 2009 Risk, ambiguity, and the dark-dependence axioms. *Am. Econ. Rev.* **99**, 385–392. (doi:10.1257/aer.99.1.385)
11. Tversky A, Kahenman D. 1983 Extensional versus intuitive reasoning: the conjunction fallacy in probability judgement. *Psychol. Rev.* **90**, 293–315. (doi:10.1037/0033-295X.90.4.293)
12. Tversky A, Shafir E. 1992 The disjunction effect in choice under uncertainty. *Psychol. Sci.* **3**, 305–309. (doi:10.1111/j.1467-9280.1992.tb00678.x)
13. Aerts D, Sozzo S, Tapia J. 2012 A quantum model for the Ellsberg and Machina paradoxes. *Lect. Notes Comput. Sci.* **7620**, 48–59. (doi:10.1007/978-3-642-35659-9_5)
14. Asano M, Ohya M, Tanaka Y, Basieva I, Khrennikov A. 2011 Quantum-like model of brain's functioning: decision making from decoherence. *J. Theor. Biol.* **281**, 56–64. (doi:10.1016/j.jtbi.2011.04.022)
15. Asano M, Ohya M, Tanaka Y, Basieva I, Khrennikov A. 2012 Quantum-like dynamics of decision-making. *Phys. A* **391**, 2083–2099. (doi:10.1016/j.physa.2011.11.042)
16. Asano M , Basieva I, Khrennikov A, Ohya M, Tanaka Y, Yamato I. 2012 Quantum-like model for the adaptive dynamics of the genetic regulation of *E. coli's* metabolism of glucose/lactose. *Syst. Synth. Biol.* **6**, 1–7. (doi:10.1007/s11693-012-9091-1)
17. Asano M, Khrennikov A, Ohya M, Tanaka Y, Yamato I. 2015 *Quantum adaptivity in biology: from genetics to cognition*. Berlin, Germany: Springer.
18. Bagarello F. 2012 *Quantum dynamics for classical systems: with applications of the number operator*. New York, NY: John Wiley.
19. Bagarello F. 2015 A quantum-like view to a generalized two players game. *Int. J. Theor. Phys.* **54**, 3612–3627. (doi:10.1007/s10773-015-2599-x)
20. Bagarello F. 2015 An operator view on alliances in politics. *SIAM J. Appl.. Math.* **75**, 564–584. (doi:10.1137/140990747)
21. Bagarello F, Haven E. 2016 First results on applying a non-linear effect formalism to alliances between political parties and buy and sell dynamics. *Phys. A* **444**, 403–414. (doi:10.1016/j.physa.2015.10.022)

22. Boyer-Kassem T, Duchene S, Guerci E. 2015 Quantum-like models cannot account for the conjunction fallacy. *Theor. Decis.* **10**, 1–32. (doi:10.1007/s11238-016-9549-9)
23. Boyer-Kassem T, Duchene S, Guerci E. 2016 Testing quantum-like models of judgment for question order effect. *Math. Soc. Sci.* **80**, 33–46. (doi:10.1016/j.mathsocsci.2016.01.001)
24. Brandenburger A. 2010 The Relationship between quantum and classical correlation in games. *Game Econ. Behav.* **69**, 175–183. (doi:10.1016/j.geb.2009.10.009)
25. Busemeyer JR, Wang Z, Townsend JT. 2006 Quantum dynamics of human decision making. *J. Math. Psychol.* **50**, 220–241. (doi:10.1016/j.jmp.2006.01.003)
26. Busemeyer JR, Pothos EM, Franco R, Trueblood J. 2011 A quantum theoretical explanation for probability judgment errors. *Psychol. Rev.* **118**, 193–218. (doi:10.1037/a0022542)
27. Busemeyer JR, Bruza PD. 2012 *Quantum models of cognition and decision*. Cambridge, UK: Cambridge University Press.
28. de Barros AJ, Suppes P. 2009 Quantum mechanics, interference, and the brain. *J. Math. Psychol.* **53**, 306–313. (doi:10.1016/j.jmp.2009.03.005)
29. de Barros AJ. 2012 Quantum-like model of behavioral response computation using neural oscillators. *Biosystems* **110**, 171–182. (doi:10.1016/j.biosystems.2012.10.002)
30. de Barros AJ, Oas G. 2014 Negative probabilities and counter-factual reasoning in quantum cognition. *Phys. Scripta* **T163**, 014008. (doi:10.1088/0031-8949/2014/T163/014008)
31. Dzhafarov EN, Kujala JV. 2014 On selective influences, marginal selectivity, and Bell/CHSH inequalities. *Top. Cogn. Sci.* **6**, 121–128. (doi:10.1111/tops.12060)
32. Haven E, Khrennikov A. 2013 *Quantum social science*. Cambridge, UK: Cambridge University Press.
33. Haven E, Khrennikov A. 2017 Statistical and subjective interpretations of probability in quantum-like models of cognition and decision making. *J. Math. Psych.* **74**, 82–91. (doi:10.1016/j.jmp.2016.02.005)
34. Khrennikov A. 2004 *Information dynamics in cognitive, psychological, social, and anomalous phenomena*. Fundamental Theories of Physics. Dordrecht, The Netherlands: Kluwer.
35. Khrennikov A. 2010 *Ubiquitous quantum structure: from psychology to finance*. Berlin, Germany: Springer.
36. Khrennikov A. 2015 Quantum version of Aumann's approach to common knowledge: sufficient conditions of impossibility to agree on disagree. *J. Math. Econ.* **60**, 89–104. (doi:10.1016/j.jmateco.2015.06.018)
37. Khrennikov A, Basieva I. 2014 Possibility to agree on disagree from quantum information and decision making. *J. Math. Psychol.* **62–63**, 1–15. (doi:10.1016/j.jmp.2014.09.003)
38. Khrennikova P. 2016 Application of quantum master equation for long-term prognosis of asset-prices. *Phys. A* **450**, 253–263. (doi:10.1016/j.physa.2015.12.135)
39. Khrennikova P, Haven E. 2016 Instability of political preferences and the role of mass media: a dynamical representation in a quantum framework. *Phil. Trans. R. Soc. A* **374**, 20150106. (doi:10.1098/rsta.2015.0106)
40. Plotnitsky A. 2014 Are quantum-mechanical-like models possible, or necessary, outside quantum physics? *Phys. Scripta* **T163**, 014011. (doi:10.1088/0031-8949/2014/T163/014011)
41. Pothos EM, Busemeyer JR. 2013 Can quantum probability provide a new direction for cognitive modeling? *Behav. Brain Sci.* **36**, 255–327. (doi:10.1017/S0140525X12001525)
42. Pothos EM, Busemeyer JR. 2009 A quantum probability explanation for violation of rational decision theory. *Proc. R. Soc. B* **276**, 2171–2178. (doi:10.1098/rspb.2009.0121)
43. Pothos EM, Perry G, Corr PJ, Matthew MR, Busemeyer JR. 2011 Understanding cooperation in the Prisoner's Dilemma game. *Pers. Indiv. Differ.* **51**, 210–215. (doi:10.1016/j.paid.2010.05.002)
44. Trueblood JS, Busemeyer JR. 2012 A quantum probability model of causal reasoning. *Front. Cogn.* **2**, 289.
45. Bratteli O, Robinson DW. 1987 *Operator algebras and quantum statistical mechanics 1*. Berlin, Germany: Springer.
46. Chiribella G, d'Ariano GM, Perinotti P. 2012 Informational axioms for quantum theory. In *Foundations of probability and physics* (eds M d'Ariano *et al.*). American Institute of Physics Conference Proceedings, vol. 6, pp. 270–279. Melville, NY.
47. Brukner C, Zeilinger A. 1999 Operationally invariant information in quantum mechanics. *Phys. Rev. Lett.* **83**, 3354–3357. (doi:10.1103/PhysRevLett.83.3354)
48. Roman P. 1965 *Advanced quantum mechanics*. Reading, MA: Addison-Wesley.

49. Barnett SM, Radmore PM. 1997 *Methods in theoretical quantum optics*. Oxford, UK: Clarendon Press.
50. Khrennikov A. 2016 *Probability and randomness: quantum versus classical*. Singapore: World Scientific Publishers.
51. Fuchs CA, Schack R. 2011 A quantum-Bayesian route to quantum-state space. *Found. Phys.* **41**, 345–356. (doi:10.1007/s10701-009-9404-8)
52. Fuchs CA, Schack R. 2013 Quantum-Bayesian coherence. *Rev. Mod. Phys.* **85**, 1693–1715. (doi:10.1103/RevModPhys.85.1693)
53. Savage LJ. 1954 *The foundations of statistics*. New York, NY: John Wiley.
54. Feynman R, Hibbs A. 1965 *Quantum mechanics and path integrals*. New York, NY: McGraw-Hill.
55. Kolmogorov. 1933.
56. Erev I, Ert E, Plonsky O, Cohen D, Cohen O. 2017 From anomalies to forecasts: toward a descriptive model of decisions under risk, under ambiguity, and from experience. *Psychol. Rev.* **124**, 369–409. (doi:10.1037/rev0000062)
57. Machina M. 1982 Expected utility analysis without the independence axiom. *Econometrica* **50**, 277–323. (doi:10.2307/1912631)
58. Machina MJ. 1983 Generalized expected utility analysis and the nature of observed violations of the independence axiom. In *Foundations of utility and risk theory with applications* (eds B Stigum, F Wenstop). Dordrecht, The Netherlands: D. Reidel Publishing Co.
59. Machina MJ. 1987 Choice under uncertainty: problems solved and insolved. *J. Econ. Perspect.* **1**, 121–154. (doi:10.1257/jep.1.1.121)

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